

A First Course in String Theory.

Second Edition.

Solutions for problems in Part I.

The following pages contain the solutions for problems to be found in Part I of the textbook *A First Course in String Theory, Second edition*. The handwritten solutions are all due to Jeffrey Goldstone. They are clear and elegant. If you distribute some solutions to students please let them know that they are due to Goldstone. The rest of the solutions have been typeset. They were mostly written by me, with some help by Stefanos Marnerides and Ian Ellwood. Some of the solutions for the newer problems are due to Alan Guth. Other's contributions are indicated when applicable.

This file contains the solutions for **all problems** in Part I of the second edition, but some edits remain to be done on the handwritten solutions concerning the numbering of the problems and the equations they cite in the book (both of which changed from the first to the second edition).

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26 May 2009

Quick Calculation 2.2.

$$\begin{aligned}a'^0 &= \gamma(a^0 - \beta a^1) \\a'^1 &= \gamma(-\beta a^0 + a^1) \\a'^2 &= a^2 \\a'^3 &= a^3\end{aligned}$$

$$\begin{aligned}b'^0 &= \gamma(b^0 - \beta b^1) \\b'^1 &= \gamma(-\beta b^0 + b^1) \\b'^2 &= b^2 \\b'^3 &= b^3\end{aligned}$$

$$\begin{aligned}b'_0 &= -\gamma(b^0 - \beta b^1) \\b'_1 &= \gamma(-\beta b^0 + b^1) \\b'_2 &= b^2 \\b'_3 &= b^3\end{aligned}$$

Then

$$\begin{aligned}a'^\mu b'_\mu &= -\gamma(a^0 - \beta a^1)\gamma(b^0 - \beta b^1) + \gamma(-\beta a^0 + a^1)\gamma(-\beta b^0 + b^1) + a^2 b_2 + a^3 b_3 \\&= \gamma^2 \left[a^0 b^0 (-1 + \beta^2) + a^1 b^1 (1 - \beta^2) \right] + a^2 b^2 + a^3 b^3 \\&= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3 \\&= a^\mu b_\mu .\end{aligned}$$

Quick Calculation 2.3.

Since we are leaving the coordinates x^2 and x^3 invariant, we only consider boosts along the x^1 axis, for which

$$x'^0 = \gamma(x^0 - \beta x^1).$$

To get

$$x'^0 = x^+ = \frac{1}{\sqrt{2}}(x^0 + x^1),$$

we would need

$$\gamma = \frac{1}{\sqrt{2}}, \quad \text{and} \quad \beta = -1,$$

which is **not** possible given that $\gamma = 1/\sqrt{1 - \beta^2}$.

More conceptually, under Lorentz transformations we must have

$$-(x'^0)^2 + (x'^1)^2 + (x'^2)^2 + (x'^3)^2 = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

If light-cone coordinates (x^+, x^-, x^2, x^3) defined a (primed) Lorenz frame we would have

$$-(x^+)^2 + (x^-)^2 + (x^2)^2 + (x^3)^2 = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 \quad (\text{not true})$$

This requires

$$-(x^+)^2 + (x^-)^2 = -(x^0)^2 + (x^1)^2 \quad (\text{not true})$$

Indeed, a short calculation shows that

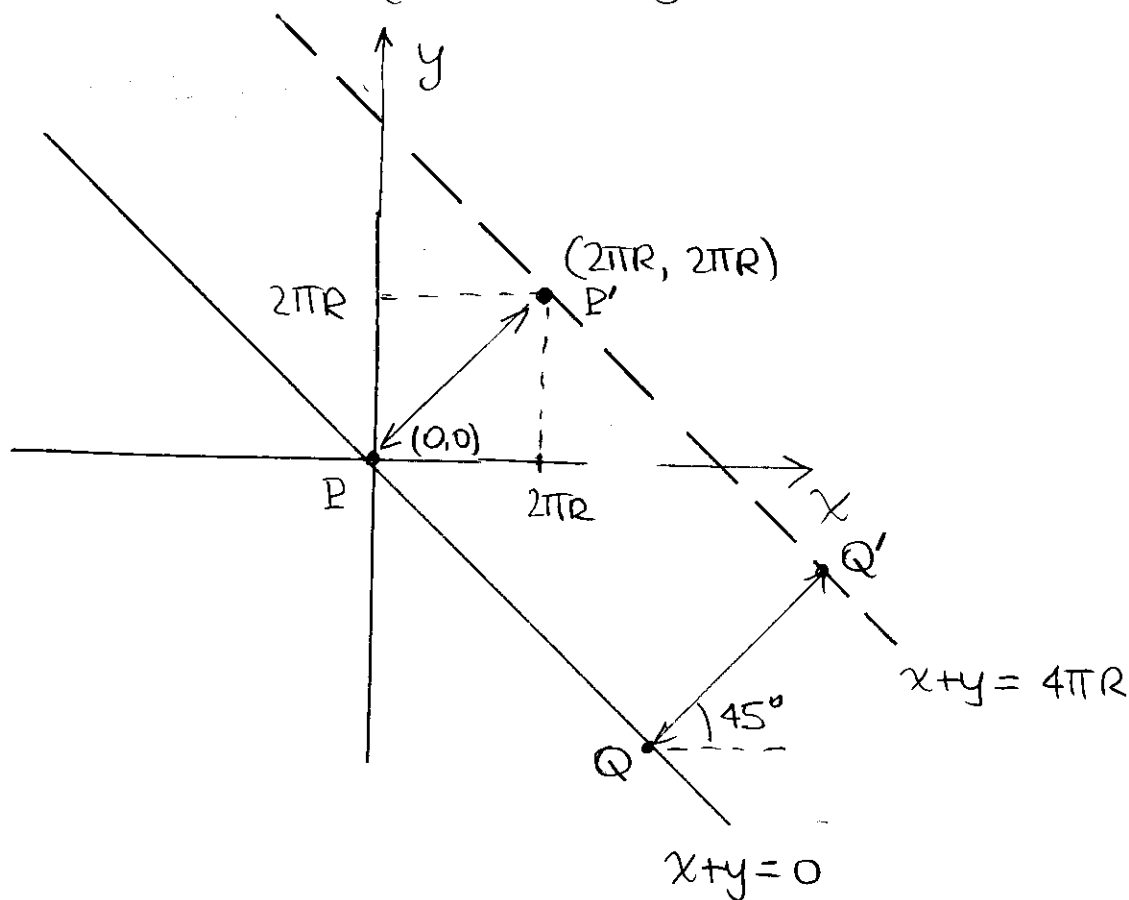
$$-(x^+)^2 + (x^-)^2 = -2x^0x^1 \neq -(x^0)^2 + (x^1)^2.$$

Light-cone frames are not Lorentz frames.

Quick Calculation 2.5

Plane (x, y) with identification

$$(x, y) \sim (x + 2\pi R, y + 2\pi R)$$



There is just one identification, note $P \sim P'$ and, more generally, $Q \sim Q'$. The lines $x+y=0$ and $x+y=4\pi R$ bound the fundamental domain $F: (x, y)$ with $0 \leq x+y < 4\pi R$.

With boundary, $\bar{F}: (x, y)$ with $0 \leq x+y \leq 4\pi R$

Identifying the two lines we get a cylinder

(B-Zwisch)

2.1. Exercises with units

(a) By definition, two charges of one esu each placed a distance 1cm from each other, repel with a force of magnitude 1 dyne. Let q denote the charge in Coulombs that is equivalent to one esu. We use the equations

$$|\vec{F}| = \frac{1}{4\pi\epsilon_0} \frac{|q_1 q_2|}{r^2} \quad \text{with} \quad \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}$$

and since 1 dyne is equivalent to 10^{-5} Newtons,

$$10^{-5} \text{N} = 8.99 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \frac{q^2}{(10^{-2} \text{m})^2} \quad \rightarrow \quad q = 3.334 \times 10^{-10} \text{C}.$$

From this we conclude that 1 esu is equivalent to 3.34×10^{-10} Coulombs, or alternatively, 1 Coulomb is equivalent to 2.998×10^9 esu.

(b) The fundamental temperature τ is defined from $\frac{1}{\tau} = \partial S / \partial E$ where S is entropy (unitless) and E is energy. So, the fundamental temperature has units of energy. The Kelvin temperature T arises from $\tau = k_B T$, so $[k_B T] = [E]$. The constant k_B can be measured once we fix the Kelvin scale. This is done by specifying arbitrarily the value for the temperature T_{tp} of the triple point of water: $T_{\text{tp}} \equiv 273.16 \text{K}$. Then k_B can be measured, say from $PV = Nk_B T$, and from measurements on a dilute gas $k_B \simeq 1.38 \times 10^{-16} \text{erg/K}$. Just as Coulombs always appear with ϵ_0 , so that Coulombs always cancel in formulae, Kelvin temperature T always appears with k_B , and K cancels.

(c) Recall that $[e] = M^{1/2} L^{3/2} T^{-1}$, $[\hbar] = M L^2 T^{-1}$, and $[c] = L T^{-1}$. Therefore $[\hbar c] = M L^3 T^{-2} = [e^2]$ and $e^2 / \hbar c$ is dimensionless. With $e = 4.802 \times 10^{-10} \text{esu}$, $\hbar = 1.504 \times 10^{-27} \text{erg} \cdot \text{s}$, and $c = 2.997 \times 10^{10} \text{cm/s}$, we find

$$\frac{e^2}{\hbar c} = 7.297 \times 10^{-3} \approx \frac{1}{137}.$$

This is the fine structure constant.

2.2. Lorentz Transformation for light cone coordinates.

(a) The nontrivial Lorentz transformations for this boost are:

$$\begin{aligned}x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0).\end{aligned}$$

We then find

$$\begin{aligned}x'^{\pm} &= \frac{x'^0 \pm x'^1}{\sqrt{2}} = \frac{\gamma}{\sqrt{2}}(x^0 - \beta x^1 \pm x^1 \mp \beta x^0) \\ &= \frac{\gamma(1 \mp \beta)}{\sqrt{2}}(x^0 \pm x^1) = \gamma(1 \mp \beta)x^{\pm},\end{aligned}$$

Therefore,

$$x'^+ = \sqrt{\frac{1-\beta}{1+\beta}} x^+, \quad x'^- = \sqrt{\frac{1+\beta}{1-\beta}} x^-, \quad x'^2 = x^2, \quad x'^3 = x^3. \quad (1)$$

The light-cone coordinates x^+ and x^- do not mix under boosts along x^1 !

(b)

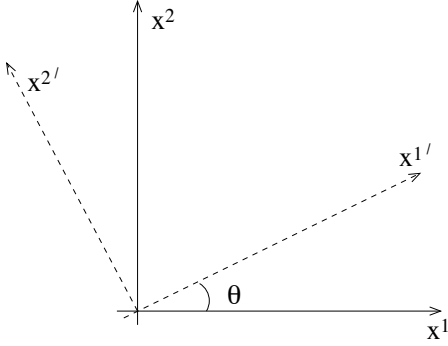


Figure 1: A rotation of coordinates.

The new, rotated coordinates are given by

$$\begin{aligned}x'^0 &= x^0 \\ x'^1 &= \cos \theta x^1 + \sin \theta x^2 \\ x'^2 &= -\sin \theta x^1 + \cos \theta x^2 \\ x'^3 &= x^3.\end{aligned}$$

After some algebra, we find

$$\begin{aligned}
x'^+ &= \frac{1}{2}(1 + \cos \theta) x^+ + \frac{1}{2}(1 - \cos \theta) x^- + \frac{\sin \theta}{\sqrt{2}} x^2, \\
x'^- &= \frac{1}{2}(1 - \cos \theta) x^+ + \frac{1}{2}(1 + \cos \theta) x^- - \frac{\sin \theta}{\sqrt{2}} x^2, \\
x'^2 &= -\frac{\sin \theta}{\sqrt{2}} x^+ + \frac{\sin \theta}{\sqrt{2}} x^- + \cos \theta x^2, \\
x'^3 &= x^3.
\end{aligned}$$

(c) For a boost with velocity parameter β along x^3 , the Lorentz transformations are:

$$\begin{aligned}
x'^0 &= \gamma(x^0 - \beta x^3) \\
x'^1 &= x^1 \\
x'^2 &= x^2 \\
x'^3 &= \gamma(x^3 - \beta x^0).
\end{aligned}$$

Therefore

$$\begin{aligned}
x'^+ &= \frac{x'^0 + x'^1}{\sqrt{2}} = \frac{\gamma(x^0 - \beta x^3) + x^1}{\sqrt{2}} \\
&= \frac{\gamma}{2}(x^+ + x^-) - \frac{\gamma\beta x^3}{\sqrt{2}} + \frac{1}{2}(x^+ - x^-).
\end{aligned}$$

Working similarly for x'^- , x'^2 , and x'^3 we obtain

$$\begin{aligned}
x'^+ &= \frac{1}{2}(\gamma + 1)x^+ + \frac{1}{2}(\gamma - 1)x^- - \frac{\gamma\beta}{\sqrt{2}}x^3, \\
x'^- &= \frac{1}{2}(\gamma - 1)x^+ + \frac{1}{2}(\gamma + 1)x^- - \frac{\gamma\beta}{\sqrt{2}}x^3, \\
x'^2 &= x^2, \\
x'^3 &= \gamma x^3 - \frac{\gamma\beta}{\sqrt{2}}(x^+ + x^-).
\end{aligned}$$

2.3 Lorentz transformations, derivatives and quantum operators.

a) $a_0 = -a^0, a_1 = a^1, a_2 = a^2, a_3 = a^3$

$$a'_0 = -a'^0 = -\gamma(a^0 - \beta a^1) = \gamma(a_0 + \beta a_1) \quad (1)$$

$$a'_1 = a'^1 = \gamma(a^1 - \beta a^0) = \gamma(a_1 + \beta a_0)$$

and $a'_2 = a_2, a'_3 = a_3$

b) Suppose we have a function $f(x^0, x^1, x^2, x^3)$ which we express as a function of x'^0, x'^1, x'^2, x'^3 by expressing x^μ as a function of x'^μ . The standard chain rule for partial differentiation says that

$$\frac{\partial f}{\partial x'^\mu} = \sum_{\nu=0}^3 \frac{\partial f}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} \quad \text{for } \mu=0,1,2,3$$

Using the summation convention and writing as an operator equation we get

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \quad (2)$$

We need x as a function of x' , the inverse of the Lorentz transformation that gives x' as a function of x . For a boost along the x^1 axis, the inverse is a boost with the opposite speed, so

$$x^0 = \gamma(x'^0 + \beta x'^1), \quad x^1 = \gamma(x'^1 + \beta x'^0) \quad (3)$$

Hence $\frac{\partial}{\partial x'^0} = \gamma\left(\frac{\partial}{\partial x^0} + \beta \frac{\partial}{\partial x^1}\right), \quad \frac{\partial}{\partial x'^1} = \gamma\left(\frac{\partial}{\partial x^1} + \beta \frac{\partial}{\partial x^0}\right) \quad (4)$

which is the same as (1) with $a_\mu = \frac{\partial}{\partial x^\mu}$.

[Note that this is consistent with the equations

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu, \quad df = \frac{\partial f}{\partial x^{\mu'}} dx^{\mu'}]$$

c) The operator for momentum $\vec{p}$ is $\frac{\hbar}{i} \vec{\nabla}$, i.e. (5)

$$p^1 = \frac{\hbar}{i} \frac{\partial}{\partial x^1}, \quad p^2 = \frac{\hbar}{i} \frac{\partial}{\partial x^2}, \quad p^3 = \frac{\hbar}{i} \frac{\partial}{\partial x^3}$$

The Schrödinger equation says $i\hbar \frac{\partial \psi}{\partial t} = H\psi$

where H is the energy operator.

Since $p^0 = \frac{E}{c}$ and $x^0 = ct$, this can be written

$$p^0 = i\hbar \frac{\partial}{\partial x^0} = -\frac{\hbar}{i} \frac{\partial}{\partial x^0} \quad (6)$$

If we write (5) and (6) in terms of p_μ , we remove the sign difference between the 0 component and the others.

$$p_\mu = \frac{\hbar}{i} \frac{\partial}{\partial x^\mu} \quad (7)$$

This is now Lorentz invariant, since we proved in (b) that $\frac{\partial}{\partial x^\mu}$ transforms like p_μ .

[This is history time-reversed. De Broglie associated momentum with wave number vector by arguing that $\vec{p} = \hbar \vec{k}$ had to go with $E = \hbar \omega$ (Planck) in order to get Lorentz invariant relations between particle and wave.]

2.4. Lorentz transformations as matrices.

- (a) If L_1 and L_2 are Lorentz transformations, then $L_1^T \eta L_1 = \eta$, and $L_2^T \eta L_2 = \eta$. Then

$$(L_1 L_2)^T \eta (L_1 L_2) = L_2^T L_1^T \eta L_1 L_2 = L_2^T \eta L_2 = \eta ,$$

so the result is shown.

- (b) Using $\eta = L^T \eta L$ we find

$$(L^{-1})^T \eta L^{-1} = (L^{-1})^T \left(L^T \eta L \right) L^{-1} = ((L^{-1})^T L^T) \eta (L L^{-1}) = \eta$$

which states that L^{-1} is a Lorentz transformation.

- (c) We are given that $L^T \eta L = \eta$, and we need to show that $L \eta L^T = \eta$. Taking the transpose of the starting equation gets us nowhere, since $(L^T \eta L)^T = L^T \eta L$. But we can get to where we want to go by multiplying both sides of the equation by $L \eta$

$$(L \eta)(L^T \eta L) = (L \eta)(\eta)$$

$$(L \eta L^T) \eta L = L$$

where we used $\eta^2 = I$ (I = identity matrix). Now if we multiply both sides on the right by $L^{-1} \eta$ have

$$(L \eta L^T) \eta L (L^{-1} \eta) = L (L^{-1} \eta)$$

which gives the desired result,

$$L \eta L^T = \eta .$$

(Solution by Alan Guth – small edits by B.Z.)

2.5 Constructing simple orbifolds.

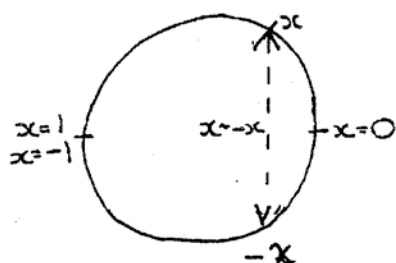
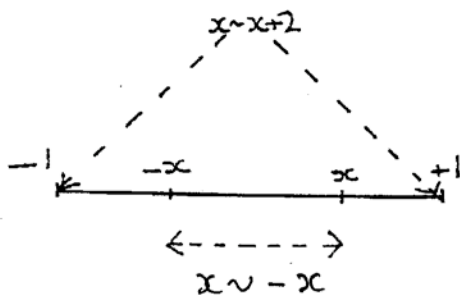
a) The $\mathbb{Z}_2$ identification $x \sim -x$ sends points in $-1 \leq x \leq 1$ into points in $-1 \leq x \leq 1$.

Two ways of picturing this are shown in the figures. If we picture S^1 as $-1 \leq x \leq 1$ with $x = \pm 1$ identified, then $x \rightarrow -x$ is reflection in the midpoint $x = 0$ and mirror image points are identified. Clearly $x = 0$ is invariant under $x \rightarrow -x$, and so is $x = 1$ which goes into $x = -1$ which is already identified with $x = 1$ under $x \sim x + 2$.

If we picture S^1 as a circle in the two-dimensional plane, $x \rightarrow -x$ is reflection in a diameter and the ends of the diameter are left invariant.

The interval $0 \leq x \leq 1$ is a fundamental domain since no two points in it are identified by $x \sim -x$ and the remaining points on the circle, $-1 < x < 0$, are obtained from $0 < x < 1$ by $x \rightarrow -x$.

This means that $S^1/\mathbb{Z}_2$ is just a closed interval, $0 \leq x \leq 1$.



b) T^2 is $-1 \leq x, y \leq 1$ with opposite edges identified
 i.e. $(-1, y) \sim (1, y)$ and $(x, -1) \sim (x, 1)$

(x, y) is invariant under $(x, y) \rightarrow (-x, -y) \iff$

- i) (x, y) is not on an edge and $x = -x, y = -y$ i.e. $x = y = 0$
 or ii) $x = 1, y = 0$
 or iii) $y = 1, x = 0$
 or iv) $x = 1, y = 1$ (all four corners of the square are identified)

Thus there are four distinct invariant points,
 $(0, 0) \quad (0, 1) \quad (1, 0) \quad (1, 1)$

As a fundamental domain we can choose the rectangle $0 \leq x \leq 1, -1 \leq y \leq 1$ together with part of its boundary.

Since $(0, y) \sim (0, -y)$ we need only keep $0 \leq y \leq 1$ on $x = 0$

Since $(1, y) \sim (-1, -y) \sim (1, -y)$ we need only keep $0 \leq y \leq 1$ on $x = 1$

Since $(x, 1) \sim (-x, -1) \sim (-x, 1)$ we need only keep $0 \leq x \leq 1$ on $y = 1$

This leaves fundamental domain shown in figure.

To present $T^2 / \mathbb{Z}_2$, take the fundamental domain and its boundary, i.e. $0 \leq x \leq 1, -1 \leq y \leq 1$ and on the boundary

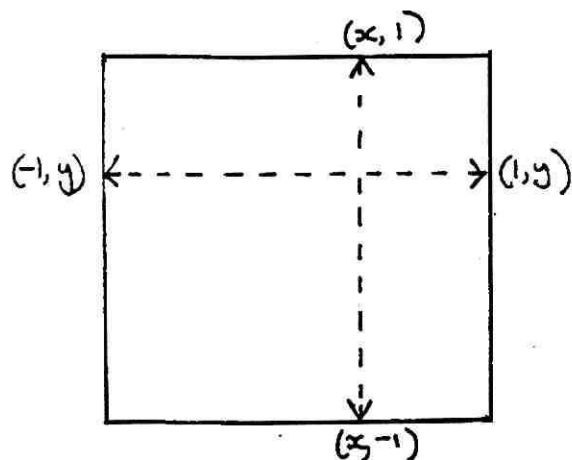
identify $(x, -1) \sim (x, 1); (0, y) \sim (0, -y); (1, y) \sim (1, -y)$

i.e. take the rectangle $0 \leq x \leq 1, -1 \leq y \leq 1$, fold it

over along $y = 0$ and sew together the edges.

Now puff up the pillowcase and we have a 2-sphere,
 S^2 .

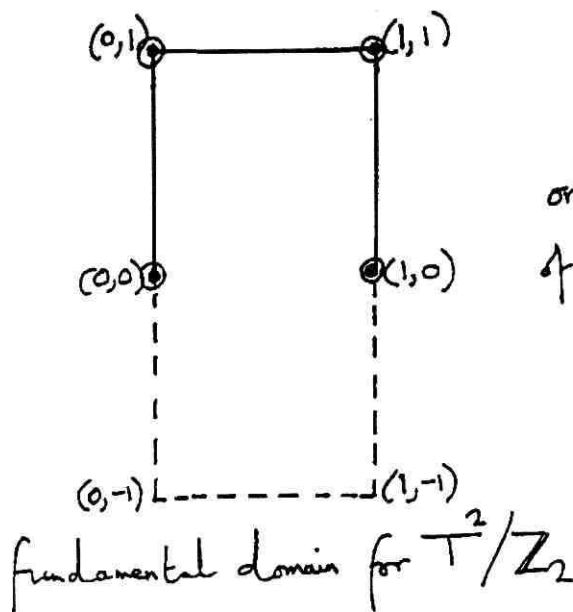
T^2



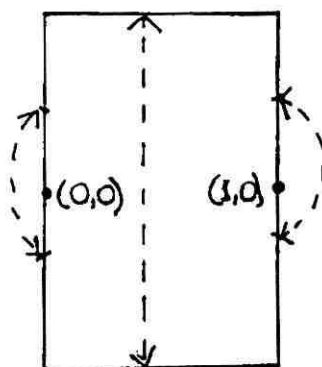
dashed lines indicate identifications

all 4 corners are identified

⊙ denotes points invariant under action of $\mathbb{Z}_2$



only solid line part of boundary included



dashed lines indicate identifications

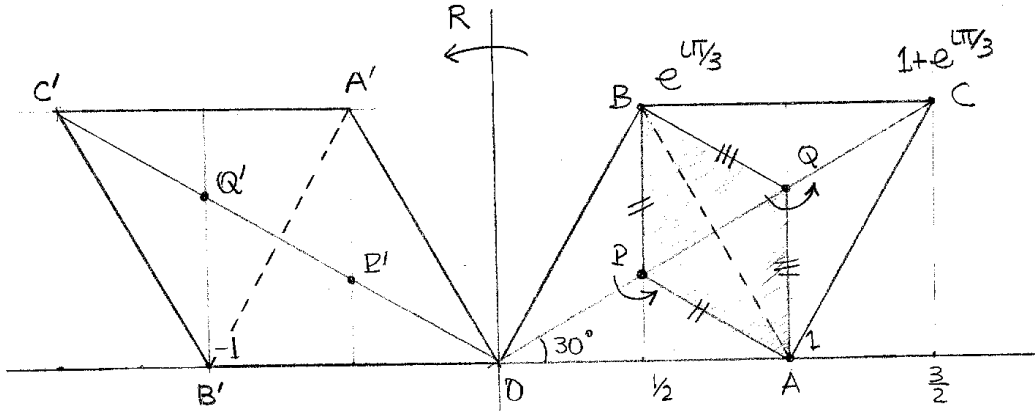
$T^2/\mathbb{Z}^2$

2.6. Constructing the $T^2/\mathbb{Z}_3$ orbifold.

(a) The last corner is at

$$z = 1 + e^{i\pi/3} = \frac{3}{2} + i\frac{\sqrt{3}}{2}.$$

This point is obtained by acting with $T_1(z) = z + 1$ on the corner at $e^{i\pi/3}$ or with $T_2(z) = z + e^{i\pi/3}$ on the corner at 1. As illustrated in the figure, T_2 identifies OA with BC and T_1 identifies OB with AC . A fundamental domain for T_1 and T_2 , together with its boundary, is provided by the parallelogram $OBCA$. Gluing the identified parallel edges we get the oblique torus.



(b) The figure shows the action of $R(z) = e^{2\pi i/3}z$ (a counterclockwise rotation by 120°) on the parallelogram $OACB$: we get the parallelogram $OA'C'B'$. Drawing the short diagonal on the parallelograms, we see that

$$\triangle OAB \rightarrow \triangle OA'B', \quad \triangle ABC \rightarrow \triangle A'B'C'$$

Now note that the unit translation T_1 takes $\triangle OA'B'$ on top of $\triangle OAB$ but with a counterclockwise rotation by 120° degrees. Furthermore two successive applications of T_1 (a translation to the right by a distance equal to two) takes $\triangle A'B'C'$ on top of $\triangle ABC$ with a counterclockwise rotation by 120° degrees as well. The rotations are indicated by the arrows around P and Q (those will turn out to be fixed points of the R action).

(c) One fixed point is clearly the origin $z = 0$ or point O ($\sim A \sim B \sim C$). It is simply invariant under R . The fixed point P is at a coordinate $z(P)$ that comes back to itself after action of R followed by a unit translation:

$$z(P) = e^{2\pi i/3}z(P) + 1 \tag{1}$$

We readily solve for $z(P)$

$$\begin{aligned} 1 &= z(P)e^{i\pi/3}(e^{-i\pi/3} - e^{i\pi/3}) \\ &= z(P)e^{i\pi/3}(-i \cdot 2 \sin \frac{\pi}{3}) \\ &= z(P)e^{i\pi/3}e^{-i\pi/2}\sqrt{3} \end{aligned}$$

We thus find

$$z(P) = \frac{1}{\sqrt{3}} e^{i\pi/6} = \frac{1}{2} + \frac{i}{2\sqrt{3}}$$

The other fixed point Q is invariant under the action of R followed by a translation by two units. Its coordinate $z(Q)$ satisfies

$$z(Q) = e^{2\pi i/3}z(Q) + 2.$$

Comparing with (1) we see that $z(Q) = 2z(P)$ solves this equation. Therefore

$$z(Q) = \frac{2}{\sqrt{3}} e^{i\pi/6} = 1 + \frac{i}{\sqrt{3}}.$$

A fundamental domain for the $\mathbb{Z}_3$ action on the triangle $\triangle OAB$ is the shaded triangle $\triangle PAB$. A fundamental domain for the $\mathbb{Z}_3$ action on the triangle $\triangle ABC$ is the shaded triangle $\triangle AQB$. So a fundamental domain for the $\mathbb{Z}_3$ action on the torus is given by the quadrilateral $AQBP$ with the edges glued as indicated by the marks on the figure. One can imagine folding this quadrilateral over the PQ segment, gluing PB to PA , and QB to QA . The result is the equilateral pillowcase.

2.7. A more general construction for cones.

The problem suggests that we experiment with low values of M and N , so let's follow that suggestion. The lowest two integers consistent with $N > M \geq 2$ are $N = 3$ and $M = 2$, so we'll try those. Then the identification becomes

$$z \sim R z, \quad \text{where } R = e^{2\pi i \left(\frac{2}{3}\right)} = e^{\frac{4\pi}{3}i}.$$

This rotation takes us more than halfway around the complex plane, so let's see what happens if we iterate it:

$$z \sim R z \implies z \sim \tilde{R} z, \quad \text{where } \tilde{R} = R^2 = e^{\frac{8\pi}{3}i} = e^{2\pi i/3}.$$

Thus the complicated identification $z \sim R z$ implies the simpler identification, $z \sim \tilde{R} z$. It is easy to see that the implication works the other way, too, since $\tilde{R}^2 = R$. Thus the identification $z \sim R z$ is equivalent to the identification $z \sim \tilde{R} z$, which is the case that we have already studied. The fundamental domain is the cone with $0 \leq \arg(z) < \frac{2\pi}{3}$. In this case the value of M was irrelevant, since the identification was the same as it would have been if M were equal to 1.

Will we find a similar simplification for the general case, with

$$R = e^{2\pi i \left(\frac{M}{N}\right)},$$

where M and N are relatively prime, with $N > M \geq 2$? Using the hint, we assume that we have found integers m and n such that

$$mM + nN = 1.$$

To make use of an equation that involves the product mM , we are motivated to consider R^m , which is then given by

$$\tilde{R} = R^m = e^{2\pi i \left(\frac{mM}{N}\right)} = e^{2\pi i \left(\frac{1-nN}{N}\right)} = e^{2\pi i \left(\frac{1}{N} - n\right)} = e^{2\pi i/N}.$$

The argument is completed as before: the identification $z \sim R z$ implies the identification $z \sim \tilde{R} z$, and the identification $z \sim \tilde{R} z$ implies $z \sim R z$, since $\tilde{R}^M = R$. So the problem reduces to understanding the identification $z \sim \tilde{R} z$, which is the cone that we have already understood.

To fully understand this problem, of course, we need to understand the hint. Good discussions of Euclid's algorithm can be found on the web at

<http://mathworld.wolfram.com/EuclideanAlgorithm.html>

and

http://en.wikipedia.org/wiki/Euclidean_algorithm,

and perhaps in other places as well. The algorithm appeared in Euclid's *Elements* in about 300 BC, making it one of the oldest known mathematical algorithms.

Suppose we have two positive integers a and b , with $b < a$, and we wish to find their greatest common divisor $\text{GCD}(a, b)$. Consider dividing a by b , obtaining an integer quotient q_1 and a nonnegative integer remainder r_1 , with $r_1 < b$. Then we can write

$$a = b q_1 + r_1 \quad \text{and} \quad r_1 = a - b q_1 .$$

From the right-hand equation one can see that any integer that divides both a and b will divide r_1 , and from the left-hand equation we can see that any integer that divides both b and r_1 will also divide a . It follows that the problem of finding $\text{GCD}(a, b)$ is equivalent to finding $\text{GCD}(b, r_1)$. The second problem is easier, since the numbers are smaller. The next step is to iterate, dividing b by r_1 . Then we can write

$$b = r_1 q_2 + r_2 \quad \text{and} \quad r_2 = b - r_1 q_2 .$$

The same arguments as before imply that $\text{GCD}(r_1, r_2) = \text{GCD}(b, r_1)$, and now $0 \leq r_2 < r_1 < b < a$. The process cannot continue forever, since a smaller integer is obtained at each iteration. The only way that the process can terminate is that for some N , we will find $r_N = 0$:

$$\begin{aligned} r_{N-3} &= r_{N-2} q_{N-1} + r_{N-1} & \text{and} & & r_{N-1} &= r_{N-3} - r_{N-2} q_{N-1} . \\ r_{N-2} &= r_{N-1} q_N & \text{and} & & 0 &= r_{N-2} - r_{N-1} q_N . \end{aligned}$$

But a remainder of zero implies that r_{N-1} divides r_{N-2} , and therefore

$$\text{GCD}(a, b) = \text{GCD}(r_{N-2}, r_{N-1}) = r_{N-1} .$$

The last nonvanishing remainder is the GCD, completing the algorithm.

To obtain the theorem used above, we note that if the numbers a and b are relatively prime, then the GCD is 1, so $r_{N-1} = 1$. The right-hand equation a few lines above then reads

$$1 = r_{N-3} - r_{N-2} q_{N-1} ,$$

so 1 is expressed as a linear combination of r 's, with integer coefficients. Using the right-hand equations from the list above, each r can be expressed in terms of earlier r 's, and ultimately one has an expansion for 1 as a linear sum of a and b , as desired.

For the numerical example suggested, $187m + 35n = 1$, the right-hand equations would look like:

$$\begin{aligned} 187 &= 5 \cdot 35 + 12 \\ 35 &= 2 \cdot 12 + 11 \\ 12 &= 11 + 1 \end{aligned}$$

so we can re-assemble

$$\begin{aligned} 1 &= 12 - 11 \\ &= (187 - 5 \cdot 35) - (35 - 2 \cdot 12) \\ &= (187 - 5 \cdot 35) - (35 - 2 \cdot (187 - 5 \cdot 35)) \\ &= 3 \cdot 187 - 16 \cdot 35. \end{aligned}$$

Thus we have a solution, with $m = 3$ and $n = -16$. The answer is not unique, because we could, for example, take $m = 3 + 35 = 38$ and $n = -16 - 187 = -203$.

Solution by A. Guth, 2/9/07

2.8. Spacetime diagrams and Lorentz transformations.

Consider the boost

$$\begin{aligned}x'^0 &= \gamma(x^0 - \beta x^1) \\x'^1 &= \gamma(x^1 - \beta x^0).\end{aligned}$$

The x'^0 axis is the set of points with $x'^1 = 0$, so it corresponds to the line $x^1 = \beta x^0$. In other words, it is a line that goes through the origin and the point $x^0 = 1, x^1 = \beta$. This point gives $x'^0 = \gamma(1 - \beta^2) > 0$.

The x'^1 axis is the set of points with $x'^0 = 0$, so it corresponds to the line $x^0 = \beta x^1$. In other words, it is a line that goes through the origin and the point $x^0 = \beta, x^1 = 1$. This point gives $x'^1 = \gamma(1 - \beta^2) > 0$.

In the figure we show the old and new axes, for $\beta > 0$ (left-side) and for $\beta < 0$ (right-side). In both cases the angle ϕ between the axes (x^0 and x'^0 , or x^1 and x'^1) has magnitude $\tan |\phi| = |\beta|$. Note that $|\phi| \leq \pi/4$.

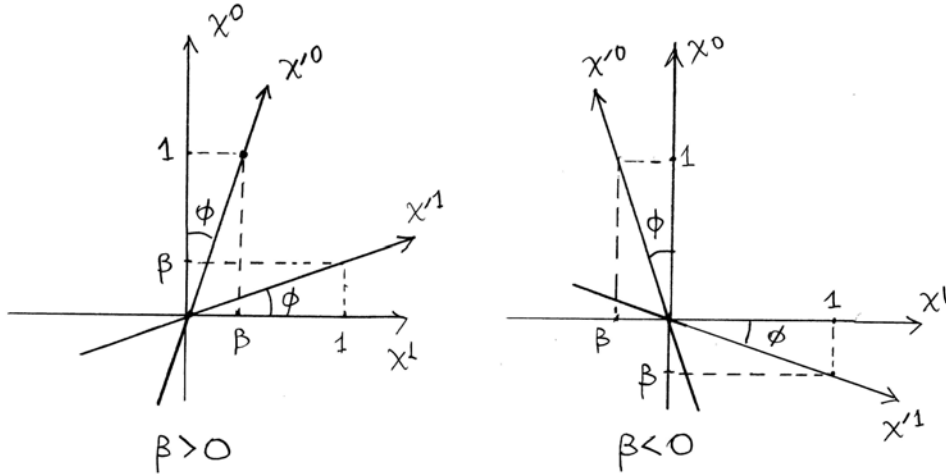


Figure 1: Oblique axes for boosts.

2.9. Light-like compactification.

$$\begin{pmatrix} x \\ ct \end{pmatrix} \sim \begin{pmatrix} x \\ ct \end{pmatrix} + 2\pi \begin{pmatrix} R \\ -R \end{pmatrix} \quad (1)$$

(a) We use the above to compute

$$\begin{aligned} x^+ &= \frac{ct + x}{\sqrt{2}} \sim \frac{1}{\sqrt{2}} (ct - 2\pi R + x + 2\pi R) = x^+ \\ x^- &= \frac{ct - x}{\sqrt{2}} \sim \frac{1}{\sqrt{2}} (ct - 2\pi R - x - 2\pi R) = x^- - 2\pi(\sqrt{2}R) \end{aligned}$$

We therefore have the light-cone identifications

$$x^+ \sim x^+, \quad x^- \sim x^- - 2\pi(\sqrt{2}R).$$

The compactification does not involve light-cone time.

(b) With the Lorentz transformations $x' = \gamma(x - \beta ct)$ and $ct' = \gamma(ct - \beta x)$, we find

$$\begin{aligned} x' &\sim \gamma(x + 2\pi R - \beta[ct - 2\pi R]) = x' + 2\pi R\gamma(1 + \beta) \\ ct' &\sim \gamma(ct - 2\pi R - \beta[x + 2\pi R]) = ct' - 2\pi R\gamma(1 + \beta) \end{aligned}$$

thus

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} \sim \begin{pmatrix} x' \\ ct' \end{pmatrix} + 2\pi \sqrt{\frac{1+\beta}{1-\beta}} \begin{pmatrix} R \\ -R \end{pmatrix}. \quad (2)$$

Comparing with (1) we learn that in the boosted frame the effective radius of compactification is increased.

Now consider the compactification

$$\begin{pmatrix} x \\ ct \end{pmatrix} \sim \begin{pmatrix} x \\ ct \end{pmatrix} + 2\pi \begin{pmatrix} \sqrt{R^2 + R_s^2} \\ -R \end{pmatrix}$$

(c) We search for a frame (x', ct')

$$\begin{aligned} x' &= \gamma(x - \beta ct) \\ ct' &= \gamma(ct - \beta x) \end{aligned}$$

in which this compactification is standard. For this we need $ct' \sim ct'$. So,

$$\begin{aligned} ct' &\sim \gamma(ct - 2\pi R - \beta[x + 2\pi\sqrt{R^2 + R_s^2}]) \\ &= ct' - \gamma(2\pi)(R + \beta\sqrt{R^2 + R_s^2}) \end{aligned}$$

Thus, we need $R + \beta\sqrt{R^2 + R_s^2} = 0$, so

$$\beta = -\frac{R}{\sqrt{R^2 + R_s^2}}. \quad (3)$$

Now examine the x' identification to find the radius:

$$\begin{aligned} x' &\sim \gamma(x + 2\pi\sqrt{R^2 + R_s^2} - \beta(ct - 2\pi R)) \\ x' &\sim x' + 2\pi\gamma(\sqrt{R^2 + R_s^2} + \beta R). \end{aligned}$$

Simplify,

$$\begin{aligned} \gamma(\sqrt{R^2 + R_s^2} + \beta R) &= \frac{1}{\sqrt{1 - \beta^2}}(\sqrt{R^2 + R_s^2} - \frac{R^2}{\sqrt{R^2 + R_s^2}}) \\ &= \frac{\sqrt{R^2 + R_s^2}}{R_s}(\frac{R^2 + R_s^2 - R^2}{\sqrt{R^2 + R_s^2}}) = R_s. \end{aligned}$$

Thus, $x' \sim x' + 2\pi R_s$. The radius of compactification is R_s . It is interesting to note that $\gamma = \sqrt{R^2 + R_s^2}/R_s$. In the limit as R_s is small, this is a very large Lorentz factor $\gamma \simeq R/R_s$.

(d) The required spacetime diagram is shown in Figure 1.

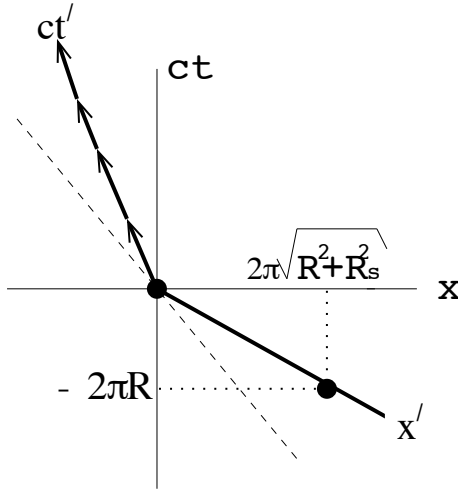


Figure 1: The heavy dots are identified so that the x' axis must go through them. The world line of the origin of the primed system is the ct' line. This origin is moving in the negative x -direction, as the sign of β indicates.

(e) Light like compactification with radius R arises by boosting a standard compactification with radius R_s with Lorentz factor $\gamma \simeq R/R_s$, in the limit as $R_s \rightarrow 0$.

2.10. A spacetime orbifold in two-dimensions.

(a) From Problem 2.2 we know that under a boost with velocity parameter β along the x^1 direction, x^+ and x^- transform as

$$x^+ \rightarrow \sqrt{\frac{1-\beta}{1+\beta}} x^+ \quad \text{and} \quad x^- \rightarrow \sqrt{\frac{1+\beta}{1-\beta}} x^-.$$

Since these positive scale factors are inverses of each other we may define

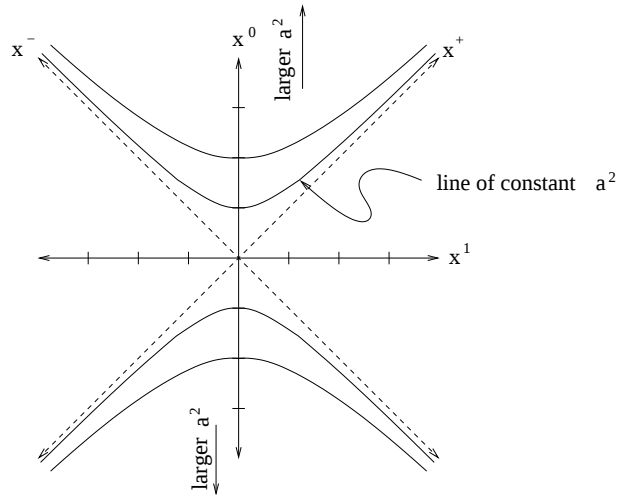
$$e^\lambda \equiv \sqrt{\frac{1+\beta}{1-\beta}}, \quad -\infty \leq \lambda \leq \infty.$$

λ is called the rapidity. Given two boosts with rapidities λ_1 and λ_2 , the rapidity of the combined transformation is clearly $\lambda_1 + \lambda_2$. The range $-\infty \leq \lambda \leq \infty$ arises because $\beta \in (-1, 1)$, so $e^\lambda \in (0, \infty)$. We can write the identification as

$$(x^+, x^-) \sim (e^{-\lambda} x^+, e^\lambda x^-).$$

For arbitrary λ the orbifold fixed point is the origin $x^+ = x^- = 0$.

(b) Below we plot curves of constant a^2 for two values of a^2 . Note that there are upper and lower branches.



Note that the quantity $x^+ x^-$ is invariant under boosts since x^+ and x^- transform get multiplied by $e^{-\lambda}$ and e^λ respectively under a Lorentz transformation. This implies that the identification only equates points on the same $x^+ x^- = a^2$ curve.

(c) We have $x^+ = a^2/x^-$ so that $dx^+ = -\left(\frac{a}{x^-}\right)^2 dx^-$. This yields

$$-ds^2 = -2dx^+dx^- = 2\left(\frac{a}{x^-}\right)^2 (dx^-)^2 > 0.$$

Thus $ds^2 < 0$ and the curve is spacelike.

(d) From part (c) we have that $-ds^2 = 2a^2\left(\frac{dx^-}{x^-}\right)^2$ so the appropriate root is

$$\sqrt{2}a \frac{dx^-}{x^-}$$

To find the invariant length of the circle we must integrate the above differential from a given x_*^- to its immediate image $e^\lambda x_*^-$. This gives an invariant length

$$\sqrt{2}a \int_{x_*^-}^{e^\lambda x_*^-} \frac{dx^-}{x^-} = \sqrt{2}a \log(x^-) \Big|_{x_*^-}^{e^\lambda x_*^-} = \sqrt{2}a \log(e^\lambda) = \sqrt{2}a\lambda.$$

2.11. Extra dimensions and statistical mechanics.

(a) The states of the system are labeled by k and l and the partition function is given by

$$Z(a, R) = \sum_{k,l} \exp \left\{ -\beta \left(\frac{\hbar^2}{2m} \right) \left[\left(\frac{k\pi}{a} \right)^2 + \left(\frac{l}{R} \right)^2 \right] \right\}.$$

Here $k = 1, \dots, \infty$, while $l = 0, 1, \dots, \infty$. Due to the degeneracy of the energy levels we must include an extra factor of 2 for each term in the l sum for $l > 0$. Defining

$$\widehat{Z}(r) \equiv \sum_{k=1}^{\infty} \exp \left\{ -\beta \left(\frac{\hbar^2}{2m} \right) \left(\frac{k\pi}{r} \right)^2 \right\},$$

we have that

$$Z(a, R) = \widehat{Z}(a) \left(1 + 2\widehat{Z}(\pi R) \right). \quad (1)$$

We need to evaluate $\widehat{Z}(r)$ for small β . In this case, the terms in the sum vary very slowly with k and many of them are nearly equal to one making the total sum very large. It is thus possible to replace the sum with an integral that runs from zero to infinity:

$$\widehat{Z}(r) \simeq \int_0^{\infty} dk \exp \left\{ -\frac{\beta \hbar^2}{2m} \left(\frac{k\pi}{r} \right)^2 \right\} = \sqrt{\frac{mr^2}{2\pi\beta\hbar^2}} \gg 1, \quad \beta \rightarrow 0. \quad (2)$$

For sufficiently small β , (2) applies both for $r = a$ and $r = 2\pi R$. So (1) gives:

$$Z(a, R) \simeq \widehat{Z}(a) \cdot 2\widehat{Z}(\pi R) \simeq \sqrt{\frac{ma^2}{2\pi\beta\hbar^2}} \times 2\sqrt{\frac{m(\pi R)^2}{2\pi\beta\hbar^2}} = \sqrt{\frac{ma^2}{2\pi\beta\hbar^2}} \sqrt{\frac{m(2\pi R)^2}{2\pi\beta\hbar^2}}.$$

This partition function is just the product of two familiar factors, one using the length a and the other using $2\pi R$. For high temperature the partition function is approximately that of a particle in a two-dimensional box with sides a and $2\pi R$.

(b) The stated regime implies that the thermal energy kT satisfies the inequalities

$$\frac{\hbar^2}{mR^2} \gg kT \gg \frac{\hbar^2}{ma^2}.$$

We can use the high-temperature result (2) for $\widehat{Z}(a)$, but not for $\widehat{Z}(\pi R)$:

$$Z(a, R) = \sqrt{\frac{ma^2}{2\pi\beta\hbar^2}} \left[1 + 2\widehat{Z}(\pi R) \right] = \sqrt{\frac{ma^2}{2\pi\beta\hbar^2}} \left[1 + 2 \exp \left(-\frac{\beta \hbar^2}{2mR^2} \right) + \dots \right]$$

The leading correction arises from the first term in the sum that defines $\widehat{Z}(\pi R)$. This is a “low-temperature” small correction; the argument of the exponential is large and negative.

3.1. Lorentz covariance for motion in electromagnetic fields

We rearrange the indices by raising the μ in equation (1) of the problem statement:

$$\frac{dp^\mu}{ds} = \frac{q}{c} F^{\mu\nu} \frac{dx_\nu}{ds}.$$

Multiplying both sides of the equation by ds/dt we find

$$\frac{dp^\mu}{dt} = \frac{q}{c} F^{\mu\nu} \frac{dx_\nu}{dt}.$$

We test this equation using $F^{\mu\nu}$, as given in the text, and $\frac{dx_\nu}{dt} = (-c, v_x, v_y, v_z)$:

$$\begin{aligned} \frac{dp^1}{dt} &= \frac{q}{c}(F^{10}(-c) + F^{12}v_y + F^{13}v_z) = qE_x + \frac{q}{c}(v_yB_z - v_zB_y) \\ &= q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)_x \quad \text{good!} \\ \frac{dp^2}{dt} &= \frac{q}{c}(F^{20}(-c) + F^{21}v_x + F^{23}v_z) = qE_y + \frac{q}{c}(v_zB_x - v_xB_z) \\ &= q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)_y \quad \text{good!} \\ \frac{dp^3}{dt} &= \frac{q}{c}(F^{30}(-c) + F^{31}v_x + F^{32}v_y) = qE_z + \frac{q}{c}(v_xB_y - v_yB_x) \\ &= q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)_z \quad \text{good!} \end{aligned}$$

The last equation is

$$\frac{dp^0}{dt} = \frac{q}{c} F^{0i} \frac{dx_i}{dt} = \frac{q}{c} \vec{E} \cdot \vec{v}.$$

With $p^0 = E/c$, it becomes

$$\frac{dE}{dt} = (q\vec{E}) \cdot \vec{v} = \text{force} \times \text{velocity}$$

The rate of change of the particle energy equals the rate at which the fields do work on the particle. The magnetic force is perpendicular to the velocity and does not do work.

3.2. Maxwell equations in four dimensions

(a) T is totally antisymmetric, so T vanishes unless all the indices are different. This yields four equations: $T_{012} = 0$, $T_{013} = 0$, $T_{023} = 0$, and $T_{123} = 0$. The first three of them give

$$\begin{aligned}
T_{012} = 0 &\rightarrow \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = 0 \\
&\quad \frac{1}{c} \frac{\partial}{\partial t} B_z + \frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x = 0 \\
&\quad \rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c} \frac{\partial B_z}{\partial t} \\
T_{013} = 0 &\rightarrow \partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} = 0 \\
&\quad \frac{1}{c} \frac{\partial}{\partial t} (-B_y) + \frac{\partial}{\partial x} E_z - \frac{\partial}{\partial z} E_x = 0 \\
&\quad \rightarrow \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c} \frac{\partial B_y}{\partial t} \\
T_{023} = 0 &\rightarrow \partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} = 0 \\
&\quad \frac{1}{c} \frac{\partial}{\partial t} (B_x) + \frac{\partial}{\partial y} E_z - \frac{\partial}{\partial z} E_y = 0 \\
&\quad \rightarrow \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c} \frac{\partial B_x}{\partial t}.
\end{aligned}$$

They are the three components of $\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$. Finally $T_{123} = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$ gives

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad \rightarrow \quad \nabla \cdot \vec{B} = 0.$$

(b) Test values for μ in the equation $\partial F^{\mu\nu} / \partial x^\nu = j^\mu / c$. For $\mu = 0$:

$$\frac{\partial F^{0i}}{\partial x^i} = \frac{j^0}{c} = \rho \quad \text{which is} \quad \nabla \cdot \vec{E} = \rho.$$

$\mu = i$: To do the 3 indices at once note that $F^{ij} = \epsilon^{ijk} B^k$ where ϵ is totally antisymmetric and $\epsilon^{123} = 1$. For example $F^{12} = \epsilon^{12k} B^k = \epsilon^{123} B^3 = B^3$. Using this, we have

$$\frac{1}{c} \frac{\partial F^{i0}}{\partial t} + \frac{\partial F^{ij}}{\partial x^j} = \frac{j^i}{c} \quad \rightarrow \quad -\frac{1}{c} \frac{\partial E^i}{\partial t} + \frac{\partial}{\partial x^j} (\epsilon^{ijk} B^k) = \frac{j^i}{c}$$

A little rearrangement gives

$$\epsilon^{ijk} \frac{\partial}{\partial x^j} B^k = \frac{j^i}{c} + \frac{1}{c} \frac{\partial E^i}{\partial t}, \quad \text{which is} \quad (\nabla \times \vec{B})_i = \left(\frac{\vec{j}}{c} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right)_i.$$

3.3. Electromagnetism in three dimensions

(a) Using equation (3.11) and noting that there are no currents nor velocity in the z direction, we have

$$B_x = B_y = 0, \quad E_z = 0, \quad j_z = 0, \quad v_z = 0.$$

Maxwell's equations then become

$$\text{Maxwell's equations down to D=3} \left\{ \begin{array}{l} \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \rho \quad \text{from } \nabla \cdot \vec{E} = \rho \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c} \frac{\partial B_z}{\partial t} \quad \text{from } \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\ \frac{\partial B_z}{\partial y} = \frac{j_x}{c} + \frac{1}{c} \frac{\partial E_x}{\partial t} \\ -\frac{\partial B_z}{\partial x} = \frac{j_y}{c} + \frac{1}{c} \frac{\partial E_y}{\partial t} \end{array} \right\} \quad \text{from } \nabla \times \vec{B} = \frac{\vec{j}}{c} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

while we get nothing from $\nabla \cdot \vec{B} = 0$. The force law gives nontrivial equations only for the x and y components:

$$\frac{dp_x}{dt} = q \left(E_x + \frac{v_y}{c} B_z \right), \quad \frac{dp_y}{dt} = q \left(E_y - \frac{v_x}{c} B_z \right).$$

(b) In three dimensions we have $A^\mu = (\Phi, A^1, A^2)$, $A_\mu = (-\Phi, A^1, A^2)$, and $j^\mu = (c\rho, j^1, j^2)$. Moreover, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, so

$$F_{0i} = \frac{1}{c} \frac{\partial A_i}{\partial t} + \frac{\partial \Phi}{\partial x^i} \equiv -E_i, \quad F_{12} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \equiv B_z \quad \text{just a name!}$$

Thus the field strength looks like

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & B_z \\ E_y & -B_z & 0 \end{pmatrix} \quad F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B_z \\ -E_y & -B_z & 0 \end{pmatrix}.$$

The above $D = 3$ field strength F can be viewed as the $D = 4$ one with $E_z = B_x = B_y = 0$. The $D = 3$ current j can be viewed as the $D = 4$ current j with $j_z = 0$. The three-dimensional Maxwell equations are therefore the truncation to $E_z = B_x = B_y = 0$ and $j_z = 0$ of the original four-dimensional ones. Those are simply the ones given in part (a), so there is no need to check anything further.

The force law

$$\frac{dp^\mu}{dt} = \frac{q}{c} F^{\mu\nu} \frac{dx_\nu}{dt}$$

gives

$$\begin{aligned} \frac{dp^x}{dt} &= q \frac{F^{10}}{c} (-c) + q \frac{F^{12}}{c} v_y = q \left(E_x + \frac{v_y}{c} B_z \right), \\ \frac{dp^y}{dt} &= q \frac{F^{20}}{c} (-c) + q \frac{F^{21}}{c} v_x = q \left(E_y - \frac{v_x}{c} B_z \right), \end{aligned}$$

so everything is as expected. Also,

$$\frac{dp^0}{dt} = \frac{qF^{0i}}{c} v_i = \frac{q}{c} \vec{E} \cdot \vec{v} \quad \Rightarrow \quad \frac{dE}{dt} = q \vec{E} \cdot \vec{v} \quad \text{familiar result!}$$

3.4. Electric fields and potentials of point charge

(a) $T_{0ij} = \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i}$. For time-independent fields $\partial_0 F_{ij} = 0$, so $T_{0ij} = 0$ implies

$$\partial_i F_{0j} - \partial_j F_{0i} = 0 \quad \rightarrow \quad \partial_i E_j - \partial_j E_i = 0. \quad (1)$$

$\vec{E} = -\nabla\Phi$ says $E_i = -\partial_i\Phi$, so condition (1) is satisfied since $\partial_i\partial_j\Phi = \partial_j\partial_i\Phi$.

(b) With d *spatial* dimensions and a point charge q at $\vec{x} = 0$, the magnitude $E(r)$ of the electric field is

$$E(r) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{q}{r^{d-1}} \quad (\text{equation (3.74)}).$$

Since $E(r) = -\frac{d\Phi}{dr}(r)$,

$$\Phi(r) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{1}{d-2} \frac{q}{r^{d-2}}$$

setting $\Phi = 0$ at $r = \infty$ for $d > 2$. Using $\Gamma(x+1) = x\Gamma(x)$,

$$\Gamma\left(\frac{d}{2}\right) = \left(\frac{d}{2} - 1\right) \Gamma\left(\frac{d}{2} - 1\right) = \frac{1}{2}(d-2) \Gamma\left(\frac{d}{2} - 1\right)$$

so we get

$$\Phi(r) = \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{\frac{d}{2}}} \frac{q}{r^{d-2}}, \quad d > 2.$$

3.5 Calculating the divergence in higher dimensions.

A thin shell of radius r and width dr inside $\mathbb{R}^d$ has volume $\text{vol}(S^{d-1}(r)) dr$. Applying the divergence theorem to this shell we find

$$(\nabla \cdot \vec{f}) \text{vol}(S^{d-1}(r)) dr = f(r+dr) \text{vol}(S^{d-1}(r+dr)) - f(r) \text{vol}(S^{d-1}(r)). \quad (1)$$

The left-hand side is the volume integral of the divergence of $\vec{f}$ over the shell. The right-hand side is the net flux of $\vec{f}$ out of the shell.

Dividing both sides of (1) by $\text{vol}(S^{d-1}(r)) dr$ and taking the limit $dr \rightarrow 0$, we obtain

$$\nabla \cdot \vec{f} = \frac{1}{\text{vol}(S^{d-1}(r))} \frac{d}{dr} \left(f(r) \text{vol}(S^{d-1}(r)) \right).$$

Since $\text{vol}(S^{d-1}(r)) \sim r^{d-1}$, with a constant of proportionality that cancels out between the numerator and the denominator, we have

$$\nabla \cdot \vec{f} = \frac{1}{r^{d-1}} \frac{d}{dr} \left(r^{d-1} f(r) \right).$$

(2)

This is the final formula. For $d = 3$ it gives

$$\nabla \cdot \vec{f} = \frac{1}{r^2} \frac{d}{dr} (r^2 f(r)) = \frac{df}{dr} + \frac{2}{r} f(r).$$

3.6. Analytic continuation for gamma functions.

We begin by breaking up the integral into two pieces

$$\Gamma(z) = \int_0^1 dt e^{-t} t^{z-1} + \int_1^\infty dt e^{-t} t^{z-1}. \quad (1)$$

The first term converges for $\Re(z) > 0$, because this is needed for the region around $t = 0$. The second term converges for any z since the problematic region of integration near $t = 0$ has been removed. Indeed, whatever the value of z , we have $e^{-t} t^{z-1} \rightarrow 0$ for $t \rightarrow \infty$ and the integration out to infinity causes no problems.

Using the expansion for e^{-t} , the first term in (1) can be written as

$$A \equiv \int_0^1 dt e^{-t} t^{z-1} = \int_0^1 dt t^{z-1} \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}. \quad (2)$$

Note that in the sum the n -th term converges for $\Re(z) > -n$. If we remove the first N terms, the remaining sum from $N+1$ to ∞ will converge for $\Re(z) > -(N+1)$. We thus rewrite A as

$$\begin{aligned} A &= \int_0^1 dt t^{z-1} \sum_{n=N+1}^{\infty} \frac{(-t)^n}{n!} + \int_0^1 dt t^{z-1} \sum_{n=0}^N \frac{(-t)^n}{n!} \\ &= \int_0^1 dt t^{z-1} \left(e^{-t} - \sum_{n=0}^N \frac{(-t)^n}{n!} \right) + \sum_{n=0}^N \frac{(-1)^n}{n!} \int_0^1 dt t^{n+z-1} \end{aligned}$$

In the second term we can perform the integration assuming that $\Re(z) > 0$. Doing this, and plugging back into (1) we find

$$\Gamma(z) = \int_0^1 dt t^{z-1} \left(e^{-t} - \sum_{n=0}^N \frac{(-t)^n}{n!} \right) + \sum_{n=0}^N \frac{(-1)^n}{n!} \frac{1}{z+n} + \int_1^\infty dt e^{-t} t^{z-1}.$$

While this was derived for $\Re(z) > 0$, the integrals converge for $\Re(z) > -N-1$, so the right-hand side provides the analytic continuation. The result is a meromorphic function, namely, a function with poles. These arise from the term

$$\sum_{n=0}^N \frac{(-1)^n}{n!} \frac{1}{z+n}, \quad (3)$$

which is a sum of simple poles at the integers $n = 0, -1, -2, \dots, N$. Since we can take N to be arbitrarily large we see that $\Gamma(z)$ has a simple pole at every integer $n \leq 0$. We read off the residue of the poles from (3)

$$\text{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!}, \quad \text{Res}_{z=0} \Gamma(z) = 1.$$

3.7. Simple quantum gravity effects are small.

(a) The standard Bohr radius

$$a_0 = \frac{\hbar^2}{m_e e^2} = 5.29 \times 10^{-9} \text{cm}, \quad \text{arises from the potential } V = -\frac{e^2}{r}.$$

In the gravitational case, the potential is $V = -Gm_e m_p / r$, so, the “gravitational” Bohr radius a_0^g is obtained by replacing e^2 with $Gm_e m_p$:

$$a_0^g = \frac{\hbar^2}{m_e (Gm_e m_p)} = \frac{\hbar^2}{m_e e^2} \left(\frac{e^2}{Gm_e m_p} \right).$$

The ratio between the “gravitational” and the conventional Bohr radius is therefore

$$\frac{a_0^g}{a_0} = \frac{e^2}{Gm_e m_p} = \frac{(4.8 \times 10^{-10})^2}{(6.67 \times 10^{-8})(0.911 \times 10^{-27})(1.672 \times 10^{-24})} \simeq 2.27 \times 10^{39}.$$

This gives

$$a_0^g = 1.20 \times 10^{31} \text{cm}.$$

Since 1 light year $\approx 9.5 \times 10^{17} \text{cm}$, $a_0^g = 1.3 \times 10^{13}$ light years !! The universe is about 10^{10} years old, so this distance is fantastically large.

(b) We introduce factors of G , c , and $\hbar$ into the formula, with exponents α, β , and γ to be determined:

$$kT = \frac{1}{8\pi M} \rightarrow kT = \frac{G^\alpha c^\beta \hbar^\gamma}{8\pi M}.$$

Since $[kT] = [E] = ML^2/T^2$, using $[G] = L^3 M^{-1} T^{-2}$, $[c] = LT^{-1}$, and $[\hbar] = ML^2 T^{-1}$, we find

$$\frac{ML^2}{T^2} = L^{3\alpha+\beta+2\gamma} M^{-\alpha+\gamma-1} T^{-2\alpha-\beta-\gamma}$$

This gives three equations for three unknowns:

$$3\alpha + \beta + 2\gamma = 2, \quad -\alpha + \gamma = 2, \quad -2\alpha - \beta - \gamma = -2.$$

The solution is $\alpha = -1, \beta = 3$, and $\gamma = 1$. As a result,

$$kT = \frac{\hbar c^3}{8\pi G M} = \left(\frac{\hbar c}{G} \right) \frac{c^2}{8\pi M} = \frac{m_{\text{P}}^2 c^2}{8\pi M},$$

where $m_{\text{P}} = 2.17 \times 10^{-5} \text{gr}$ is the Planck mass. For a mass with million solar masses $M = 10^6 M_{\text{sun}} \approx 2 \times 10^{39} \text{gr}$,

$$kT = \frac{(2.17 \times 10^{-5})^2 (3 \times 10^{10})^2}{8\pi \cdot 2 \times 10^{39}} = 8.43 \times 10^{-30} \text{eV} \implies T \approx 6.1 \times 10^{-14} \text{K}.$$

This is much colder than the nanokelvins attained by Ketterle *et al.* in Bose-Einstein condensation experiments. For a room temperature black hole, $T \simeq 300$ K, and we find

$$M = \frac{6.1 \times 10^{-14} \text{ K}}{300 \text{ K}} 2 \times 10^{39} \text{ gr} \approx 4.1 \times 10^{20} \text{ kg},$$

which is approximately $6.8 \times 10^{-5} M_{\text{earth}}$ or $5.6 \times 10^{-3} M_{\text{moon}}$.

3.8. Vacuum energy and an associated length scale.

In the formula

$$\ell_{\text{vac}} = \rho_{\text{vac}}^{\alpha} \hbar^{\beta} c^{\gamma} \quad (1)$$

we note that the units must agree:

$$L = \left(\frac{M}{L^3}\right)^{\alpha} \left(\frac{ML^2}{T}\right)^{\beta} \left(\frac{L}{T}\right)^{\gamma}$$

This gives three equations, for matching powers on M , L , and T :

$$\begin{aligned} \alpha + \beta &= 0 \\ -3\alpha + 2\beta + \gamma &= 1 \\ -\beta - \gamma &= 0 \end{aligned}$$

The solution is

$$\alpha = -\frac{1}{4}, \quad \beta = \frac{1}{4}, \quad \gamma = -\frac{1}{4}. \quad (2)$$

This means that (1) gives

$$\boxed{\ell_{\text{vac}} = \left(\frac{\hbar}{c\rho_{\text{vac}}}\right)^{1/4}}. \quad (3)$$

With $\rho_{\text{vac}} = 7.7 \times 10^{-27} \text{ kg/m}^3$ we find

$$\ell_{\text{vac}} = 8.22 \times 10^{-5} \text{ m} = 82.2 \mu\text{m}.$$

It is interesting that this length scale is roughly the one at which sub-millimeter tests of gravitation are being done.

3.9. Planetary motion in four and higher dimensions.

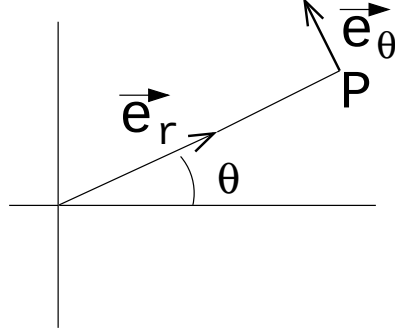


Figure 1: Motion in the plane described with polar coordinates.

We begin by deriving the usual results for effective potentials in a central force field. Using time-dependent unit vectors $\vec{e}_r(t)$ and $\vec{e}_\theta(t)$ in the radial and in the tangential direction we have:

$$\begin{aligned}\vec{x}(t) &= r(t)\vec{e}_r(t) \\ \dot{\vec{x}}(t) &= \dot{r}\vec{e}_r(t) + r\dot{\theta}\vec{e}_\theta(t) \\ \ddot{\vec{x}}(t) &= [\ddot{r} - r\dot{\theta}^2]\vec{e}_r + [2\dot{r}\dot{\theta} + r\ddot{\theta}]\vec{e}_\theta\end{aligned}$$

If the force is central (along $\vec{e}_r$) we have

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0.$$

The constancy of $r^2\dot{\theta}$ implies the constancy of the angular momentum $L = mr^2\dot{\theta}$. We thus have

$$\dot{\theta} = \frac{L}{mr^2}, \quad (1)$$

with L a constant of motion. Let $\vec{F}$ be the gravitational (central) force associated to the potential energy V_{gr} :

$$\vec{F} = -\frac{\partial V_{\text{gr}}}{\partial r} \vec{e}_r = m[\ddot{r} - r\dot{\theta}^2]\vec{e}_r.$$

Together with (1) this equation gives

$$\ddot{r} - \frac{L^2}{m^2 r^3} = -\frac{1}{m} \frac{\partial V_{\text{gr}}}{\partial r} \quad \rightarrow \quad \ddot{r} = -\frac{\partial}{\partial r} \underbrace{\left(\frac{V_{\text{gr}}}{m} + \frac{L^2}{2m^2 r^2} \right)}_{\text{effective potential}}.$$

This completes our derivation of the effective potential. Noting that the gravitational potential energy V_{gr} divided by the mass is simply the gravitational potential V_g , we have the familiar result

$$V_{\text{eff}} = V_g + \frac{L^2}{2m^2r^2} . \quad (2)$$

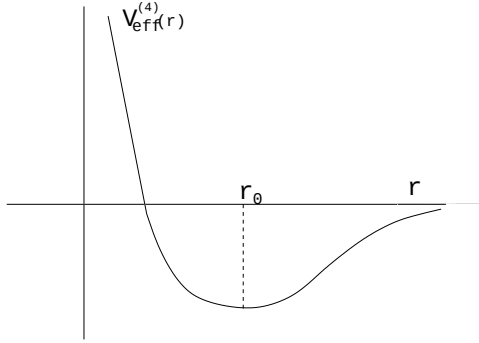


Figure 2: Effective potential for four-dimensional planetary motion.

In four dimensions

$$V_{\text{eff}}^{(4)} = -\frac{GM}{r} + \frac{L^2}{2m^2r^2} .$$

As seen by the shape of the potential in Figure 2, the radial motion is stable around the equilibrium position r_0 . Indeed

$$\left. \frac{\partial V_{\text{eff}}}{\partial r} \right|_{r_0} = +\frac{GM}{r_0^2} - \frac{L^2}{m^2r_0^3} = 0 \quad \Rightarrow \quad r_0 = \frac{L^2}{GMm^2} \quad \text{orbit radius} .$$

Computing the second derivative of the potential at the critical point we confirm it is a minimum:

$$\left. \frac{\partial^2 V_{\text{eff}}}{\partial r^2} \right|_{r_0} = -\frac{2GM}{r_0^3} + \frac{3L^2}{m^2r_0^4} = -\frac{2GM}{r_0^3} + \frac{3GM}{r_0^3} = \frac{GM}{r_0^3} > 0 \quad \longrightarrow \quad \text{stable!}$$

Our familiar world has the additional property that non-circular *closed* orbits exist, but we can't see that from the effective potential.

In five dimensions

$$V_{\text{eff}}^{(5)} = -\frac{\alpha G^{(5)} M}{2r^2} + \frac{L^2}{2m^2r^2} ,$$

with $\alpha > 0$, a dimensionless constant that can be determined using Gauss' law. Note that both terms have the same r dependence! There is a critical L_0 which makes $V_{\text{eff}}^{(5)}$ vanish:

$$L_0^2 = \alpha G^{(5)} M m^2 .$$

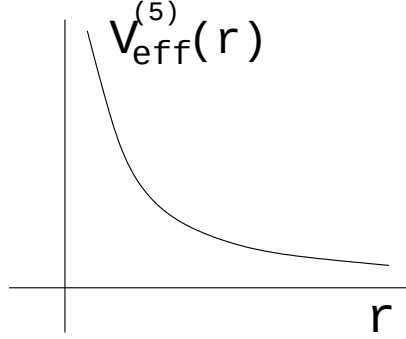


Figure 3: Five spacetime dimensions and $L > L_0$. Planet will go out to $r \rightarrow \infty$

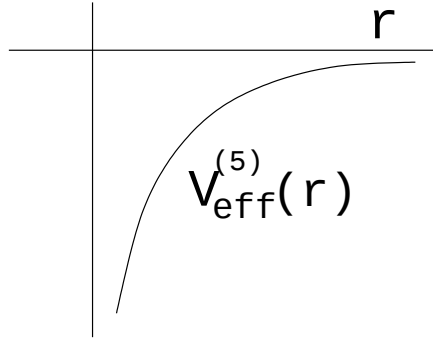


Figure 4: Five spacetime dimensions and $L < L_0$. Planet will fall into the star at $r = 0$.

In terms of L_0 the potential is

$$V_{\text{eff}}^{(5)} = \frac{1}{2m^2 r^2} (L^2 - L_0^2).$$

For $L > L_0$, see Figure 3. For $L < L_0$, see Figure 4. The orbits are unstable. They can exist for any radius if $L = L_0$ exactly, but any small perturbation that changes the angular momentum will either make the planet spiral in or out.

For spacetime dimensions $D > 5$

$$V_{\text{eff}}^{(D)} = -\alpha^{(D)} \frac{G^{(D)} M}{r^{D-3}} + \frac{L^2}{2m^2 r^2}.$$

Since $D - 3 > 2$, $V_{\text{eff}}^{(D)}$ is as in Figure 5. An unstable circular orbit is possible with an L -dependent radius r_0 .

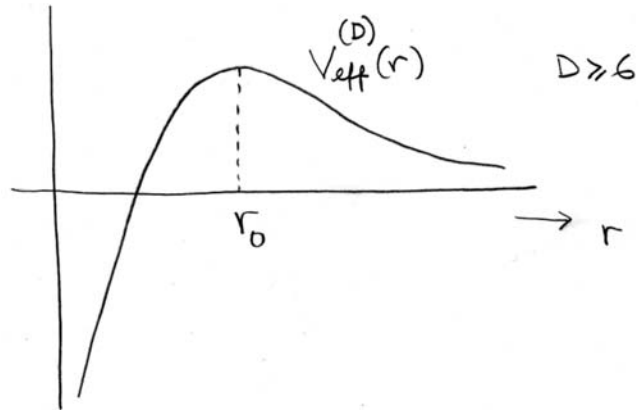


Figure 5: Effective potential for $D \geq 6$ spacetime dimensions.

3.10. Gravitational field of a point mass in compactified five-dimensional world.

(a) We examine the gravitational equation

$$\nabla^2 V_g^{(5)} = 4\pi G^{(5)} \rho_m ,$$

for a point particle of mass M fixed at the origin $r = \sqrt{x^2 + y^2 + z^2 + w^2} = 0$ of a five dimensional spacetime. Consider a four-dimensional ball $B^4(r)$ that contains the mass. The boundary of this ball is the 3-sphere $S^3(r)$. Integrating the above equation over $B^4(r)$:

$$\int_{B^4(r)} d(\text{Vol}) \nabla \cdot \nabla V_g^{(5)} = 4\pi G^{(5)} M .$$

Using the divergence theorem the left-hand side can be written as

$$\int_{S^3(r)} d(\text{Vol}) \text{Flux of } \nabla V_g^{(5)} = \int_{S^3(r)} d(\text{Vol}) \frac{\partial V_g^{(5)}}{\partial r} = \text{Vol}(S^3(r)) \cdot \frac{\partial V_g^{(5)}}{\partial r}(r)$$

because $\nabla V_g^{(5)}$ is radial and of constant magnitude over the three sphere. Since $\text{Vol}(S_3(r)) = 2\pi^2 r^3$ we get

$$2\pi^2 r^3 \frac{\partial V_g^{(5)}}{\partial r} = 4\pi G^{(5)} M \quad \rightarrow \quad \frac{\partial V_g^{(5)}}{\partial r} = \frac{2}{\pi} \frac{G^{(5)} M}{r^3} .$$

Integrating, with zero potential at infinity, we find

$$V_g^{(5)}(r) = -\frac{1}{\pi} \frac{G^{(5)} M}{r^2} . \tag{1}$$

(b) We consider the compactification $w \sim w + 2\pi a$ of the w direction into a circle of radius a (see Figure 1). Note that the potential at the point P with coordinates $(x, y, z, 0)$ cannot just be

$$V_g^{(5)}(x, y, z, 0) = -\frac{1}{\pi} \frac{G^{(5)} M}{x^2 + y^2 + z^2} \quad \textbf{not true}$$

as if we could use (1) when the space is compactified. If (1) could be used directly, for a point P' displaced vertically from P by $2\pi a$ we would also write

$$V_g^{(5)}(x, y, z, 2\pi a) = -\frac{1}{\pi} \frac{G^{(5)} M}{x^2 + y^2 + z^2 + (2\pi a)^2} \quad \textbf{not true} .$$

But then, the potentials at P and P' would be different, in contradiction with the compactification statement that P and P' are the same point.

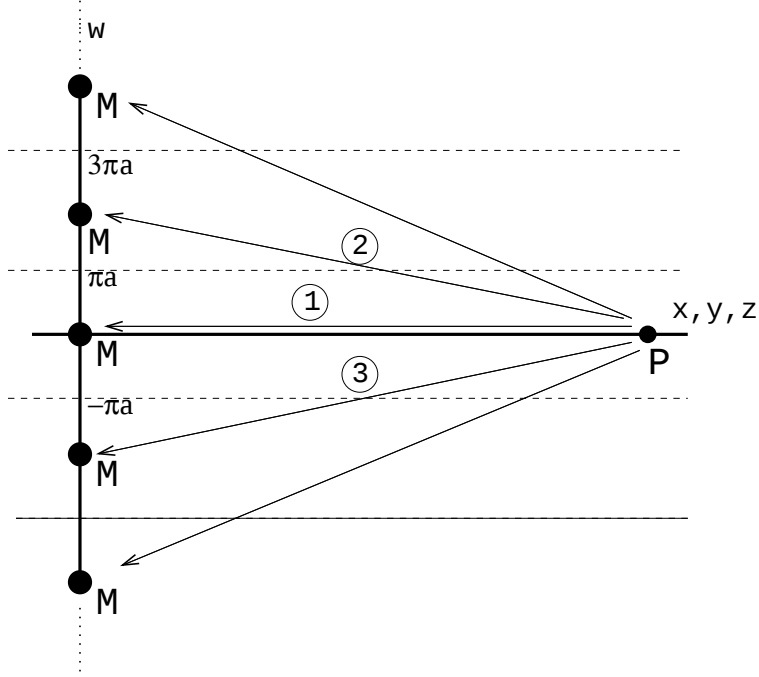


Figure 1:

To implement the w periodicity implied by the compactification we must introduce a periodic array of masses; a mass at each $w = 2\pi na$, for every integer $n \in \mathbb{Z}$. This is an infinite number of masses! Note that with periodic masses the potential will also be periodic. As shown in Figure 2, the observer actually “sees” the additional masses because the light rays wrap around the cylinder. Adding the contributions of each mass using (1), we find the potential:

$$V_g^{(5)}(x, y, z, 0) = -\frac{1}{\pi} G^{(5)} M \sum_{n=-\infty}^{\infty} \frac{1}{x^2 + y^2 + z^2 + (2\pi na)^2} = -\frac{1}{\pi} G^{(5)} M \sum_{n=-\infty}^{\infty} \frac{1}{R^2 + (2\pi na)^2}$$

where $R^2 \equiv x^2 + y^2 + z^2$. We thus have

$$V_g^{(5)}(x, y, z, 0) = -\frac{1}{\pi} \frac{G^{(5)} M}{R^2} \sum_{n=-\infty}^{\infty} \frac{1}{1 + (2\pi \frac{a}{R} n)^2} \quad (2)$$

(c) Let us evaluate the above $V_g^{(5)}$ when $\frac{a}{R} \ll 1$. In this case the terms in the sum are slowly varying and

$$\sum_{n=-\infty}^{\infty} \frac{1}{1 + (2\pi \frac{a}{R} n)^2} \cong \int_{-\infty}^{\infty} \frac{dn}{1 + (2\pi \frac{a}{R} n)^2}.$$

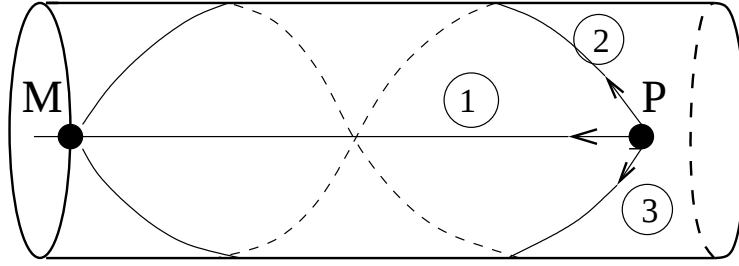


Figure 2:

Letting $n = \frac{R}{2\pi a}x$,

$$\sum_{n=-\infty}^{\infty} \frac{1}{1 + (2\pi \frac{a}{R}n)^2} \cong \frac{R}{2\pi a} \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \frac{R}{2a}$$

Back in (2) we get

$$V(x, y, z, 0) \cong -\frac{G^{(5)}M}{\pi} \frac{1}{R^2} \frac{R}{2a} = -\left(\frac{G^{(5)}}{2\pi a}\right) \frac{M}{R},$$

that is, $V \cong -\frac{GM}{R}$ with $G = \frac{G^{(5)}}{2\pi a}$, which is exactly what we wanted to show!

3.11 Exact answer for gravitational potential.

(a) Using the cited identity with $x = 2a/R$, the potential in equation (2) of Problem 3.10 becomes:

$$V_g^{(5)}(x, y, z, 0) = -\frac{G^{(5)}M}{\pi} \frac{1}{R^2} \frac{R}{2a} \coth \left[\frac{R}{2a} \right] = -\left(\frac{G^{(5)}}{2\pi a} \right) \frac{M}{R} \coth \left[\frac{R}{2a} \right] \quad (\text{exact}).$$

(b) We can do an asymptotic expansion of $\coth \left[\frac{R}{2a} \right]$ for $R \gg a$:

$$\begin{aligned} \coth \left[\frac{R}{2a} \right] &= \frac{\cosh \left[\frac{R}{2a} \right]}{\sinh \left[\frac{R}{2a} \right]} = \frac{e^{\frac{R}{2a}} + e^{-\frac{R}{2a}}}{e^{\frac{R}{2a}} - e^{-\frac{R}{2a}}} \\ &= \frac{1 + e^{-\frac{R}{a}}}{1 - e^{-\frac{R}{a}}} = 1 + \frac{2e^{-\frac{R}{a}}}{1 - e^{-\frac{R}{a}}} \\ &= 1 + 2 \left(e^{-\frac{R}{a}} + e^{-\frac{2R}{a}} + \dots \right). \end{aligned}$$

So the potential with its first leading correction is

$$V_g^{(5)} = -\left(\frac{G^{(5)}}{2\pi a} \right) \frac{M}{R} \left(1 + 2e^{-\frac{R}{a}} + \mathcal{O}(e^{-\frac{2R}{a}}) \right), \quad a \ll R.$$

Correction is 1% when $2e^{-\frac{R}{a}} = \frac{1}{100}$. This gives $\frac{R}{a} = \ln 200 \simeq 5.3$.

(c) When $R \ll a$, we are near the mass. For small x , $\coth(x) \simeq \frac{1}{x} + \frac{x}{3} + \mathcal{O}(x^3)$. So we have

$$\coth \left[\frac{R}{2a} \right] \simeq \frac{2a}{R} + \frac{R}{6a} + \dots = \frac{2a}{R} \left(1 + \frac{R^2}{12a^2} + \dots \right).$$

We then find

$$V_g^{(5)} \simeq -\frac{G^{(5)}}{2\pi a} \frac{M}{R} \frac{2a}{R} \left(1 + \frac{R^2}{12a^2} + \dots \right) = -\frac{G^{(5)}M}{\pi R^2} \left(1 + \frac{R^2}{12a^2} + \dots \right), \quad R \ll a.$$

The leading term gives the 5-dimensional answer (see equation (1) of Problem 3.10). This makes sense; very near the mass the effects of compactification must be negligible.

4.1 Consistency of small transverse oscillations

The horizontal force pulling to the left at x is

$T_0 \cos \theta$ where $\tan \theta = \frac{dy}{dx} \ll 1$ so it is

approximately $T_0 - \frac{1}{2} T_0 \left(\frac{dy}{dx} \right)^2$.

At $x+dx$ the horizontal force to the right is

$T_0 - \frac{1}{2} T_0 \left(\frac{dy}{dx} + \frac{d^2y}{dx^2} dx \right)^2$ so the difference is

$dF_h = -T_0 \frac{dy}{dx} \frac{d^2y}{dx^2} dx$. Compare this to

$dF_v = T_0 \frac{d^2y}{dx^2} dx$. The ratio is $\frac{dy}{dx}$ which for

small amplitude waves is $\ll 1$.

4.2. Longitudinal waves on strings.

Let $\eta(x, t)$ denote the longitudinal displacement at x , for any time t . One may then ask, what is the tension $T(x, t)$? Given the definition of the coefficient of tension, we can write

$$T(x, t) = T_0 + \tau_0 \frac{\Delta L}{L}.$$

To evaluate the second term on the right-hand side, we look at a small piece of parameter space from $x - \epsilon$ to $x + \epsilon$. The corresponding piece of static string has length $L = 2\epsilon$, and it is stretched $\Delta L = \eta(x + \epsilon, t) - \eta(x - \epsilon, t)$. So

$$\frac{\Delta L}{L} = \frac{1}{2\epsilon} \left(\eta(x + \epsilon, t) - \eta(x - \epsilon, t) \right) \simeq \frac{\partial \eta}{\partial x}, \quad \text{when } \epsilon \rightarrow 0.$$

As a result, we find

$$T(x, t) = T_0 + \tau_0 \frac{\partial \eta}{\partial x}. \tag{1}$$

This gives the tension at any point on the string. Now consider the piece of string associated with the parameter space interval from x to $x + dx$. The mass of this piece of string is $\mu_0 dx$, its average acceleration is $\frac{\partial^2 \eta}{\partial t^2}$, and the net force acting on it is $T(x + dx) - T(x)$. So, Newton's law gives

$$(\mu_0 dx) \frac{\partial^2 \eta}{\partial t^2} = T(x + dx) - T(x) = \frac{\partial T}{\partial x} dx.$$

Cancelling the common factor of dx and using (1), we find

$$\mu_0 \frac{\partial^2 \eta}{\partial t^2} = \tau_0 \frac{\partial^2 \eta}{\partial x^2}.$$

This takes the form of the familiar wave equation for transverse oscillations, with the tension T_0 replaced by the coefficient of tension τ_0 . The velocity v of the longitudinal waves is $v = \sqrt{\tau_0/\mu_0}$. You can also prove that the tension $T(x, t)$ satisfies exactly the same wave equation.

4.3 A configuration with two joined strings.

The equation for the oscillation frequencies breaks into two equations

$$\begin{aligned}\frac{d^2 y_1}{dx^2} + \frac{\mu_1}{T_0} \omega^2 y_1(x) &= 0, \quad \text{for } 0 \leq x \leq a, \\ \frac{d^2 y_2}{dx^2} + \frac{\mu_2}{T_0} \omega^2 y_2(x) &= 0, \quad \text{for } a \leq x \leq 2a.\end{aligned}$$

Here $y_1(x) = y(x)$ for $x \in [0, a]$ and $y_2(x) = y(x)$ for $x \in [a, 2a]$. We let

$$\lambda_1^2 = \frac{\mu_1}{T_0} \omega^2 \quad \text{and} \quad \lambda_2^2 = \frac{\mu_2}{T_0} \omega^2.$$

With this notation, the two equations take similar form

$$\frac{d^2 y_i}{dx^2} + \lambda_i^2 y_i(x) = 0, \quad \lambda_i = \sqrt{\frac{\mu_i}{T_0}} \omega, \quad i = 1, 2.$$

Since $y(x=0) = y(x=2a) = 0$ we write

$$y_1(x) = A \sin(\lambda_1 x), \quad \text{and} \quad y_2(x) = B \sin(\lambda_2(2a - x)),$$

which satisfy the differential equation and the boundary conditions at $x = 0$ and $x = 2a$.

(a) Since the string does not break at $x = a$, we must have $y_1(a) = y_2(a)$. Since there is no mass at the joining point the slopes must agree too: $y_1'(a) = y_2'(a)$.

(b) Applying the boundary conditions in (a) we find

$$\begin{aligned}A \sin \lambda_1 a &= B \sin \lambda_2 a, \\ A \lambda_1 \cos \lambda_1 a &= -B \lambda_2 \cos \lambda_2 a.\end{aligned}$$

These two equations can be written as

$$\begin{pmatrix} \sin \lambda_1 a & -\sin \lambda_2 a \\ \lambda_1 \cos \lambda_1 a & \lambda_2 \cos \lambda_2 a \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0.$$

Since we need solutions where both A and B are not zero, the determinant of the above matrix must vanish:

$$\lambda_2 \sin \lambda_1 a \cos \lambda_2 a + \lambda_1 \cos \lambda_1 a \sin \lambda_2 a = 0. \quad (1)$$

This equation can be used to find the oscillation frequencies. As a check, assume $\mu_1 = \mu_2 = \mu_0$, in which case $\lambda_1 = \lambda_2 \equiv \lambda$. The condition becomes $\sin(2\lambda a) = 0$, which gives $\lambda(2a) = n\pi$. This gives the familiar frequencies.

(c) With $\mu_1 = \mu_0$ and $\mu_2 = 2\mu_0$ we write

$$\lambda_1 = \sqrt{\frac{\mu_0}{T_0}} \omega \equiv \lambda \quad \text{and} \quad \lambda_2 = \sqrt{\frac{2\mu_0}{T_0}} \omega = \sqrt{2}\lambda.$$

Equation (1) then gives

$$\sqrt{2} \sin \lambda a \cos \sqrt{2}\lambda a + \cos \lambda a \sin \sqrt{2}\lambda a = 0.$$

Solving this equation numerically we find that the lowest root is $\lambda a \simeq 1.26567$. This gives the lowest frequency

$$\omega_L = \sqrt{\frac{T_0}{\mu_0}} \frac{1.26567}{a} = \sqrt{\frac{T_0}{\mu_0}} \frac{2.53134}{(2a)}.$$

Note that ω_L is bounded as

$$\sqrt{\frac{T_0}{\mu_0}} \frac{\pi/\sqrt{2}}{(2a)} < \omega_L < \sqrt{\frac{T_0}{\mu_0}} \frac{\pi}{(2a)}.$$

The lower and upper bounds are the lowest frequencies of strings of constant mass density $2\mu_0$ and μ_0 , respectively ($\pi/\sqrt{2} \simeq 2.22144$).

4.4 Evolving an initial string configuration

a) The most general solution of the wave equation is

$$y(t, x) = h_+(x - v_0 t) + h_-(x + v_0 t)$$

h_+ and h_- are each functions of a single variable. A solution with $h_- = 0$ has $y(0, x) = h_+(x)$ and this shape then moves towards $x > 0$ with constant speed v_0 . Similarly a solution with $h_+ = 0$ moves towards $x < 0$ without changing shape.

We can give y and $\frac{\partial y}{\partial t}$ any values at $t = 0$ by choosing h_+ and h_- .

All this applies to waves on an infinite string. We are interested in standing waves on $0 < x < a$. The initial conditions determine $h_+(x)$ and $h_-(x)$ only for $0 < x < a$.

The idea is to choose $h_+(x)$ for $x < 0$ and $h_-(x)$ for $x > a$ so that as the travelling waves $h_{\pm}(x \mp v_0 t)$ move into the region $0 < x < a$ they always exactly cancel at $x = 0$ and $x = a$. In fact I will also find the solution for $t < 0$ that becomes the solution for $t > 0$, which means

I will determine $h_+(u)$ and $h_-(u)$ for $-\infty < u < \infty$ [u is just a dummy variable that stands for the argument of the functions $h_{\pm}$. When we want to know the value of $y(t, x)$ we plug in $u = x - v_0 t$ to our table of $h_+(u)$ and $u = x + v_0 t$ to our table of $h_-(u)$]

We want $y(t, 0) = 0$ for all t

$$\text{i.e. } h_+(-v_0 t) + h_- (+v_0 t) = 0 \text{ for all } t$$

i.e. $h_+(u) = -h_-(-u)$ for all u (2.1)

We also want $y(t, a) = 0$ for all t

i.e. $h_+(a - v_0 t) = -h_-(a + v_0 t)$ for all t

From (2.1), $-h_-(a + v_0 t) = h_+(-a - v_0 t)$

so $h_+(-a - v_0 t) = h_+(a - v_0 t)$ for all t

Put $-a - v_0 t = u$. Then this says that

$h_+(u) = h_+(u + 2a)$ for all u (2.2)

If h_+ and h_- satisfy the conditions (2.1), (2.2) the Dirichlet conditions $y = 0$ at $x = 0$ and $x = a$ are satisfied by $y(t, x)$ for all t .

[You can convince yourself this is true by drawing pictures of $y(t, x)$ at successive times t rather than by the algebraic relations above]

b) Now we look at $t = 0$. $y(0, x) = 0$ for $0 < x < a$

so we need $h_+(x) + h_-(x) = 0$, $0 < x < a$

Combining this with (2.1) to get a condition on h_+ gives

$h_+(x) = h_+(-x)$ for $0 < x < a$ and so

for $-a < x < a$. Combining this with (2.2) tells us

that $h_+(u)$ is an even function i.e. $h_+(-u) = h_+(u)$ and has period $2a$ (2.3)

$$\frac{\partial y}{\partial t} = -v_0 h'_+(x - v_0 t) + v_0 h'_-(x + v_0 t)$$

$$\left[h'_+(u) = \frac{dh_+(u)}{du} \right] \quad \text{so at } t=0$$

$$-h'_+(x) + h'_-(x) = \frac{x}{a} \left(1 - \frac{x}{a} \right) \quad 0 < x < a \quad (2.4)$$

$$\text{From (2.1), } h'_+(u) = h'_-(-u)$$

and from (2.3), h'_+ is an odd function i.e. $h'_+(-u) = -h'_+(u)$

so $h'_-(x) = h'_+(-x) = -h'_+(x)$ and (2.4) becomes

$$h'_+(x) = -\frac{1}{2} \frac{x}{a} \left(1 - \frac{x}{a} \right) \quad 0 < x < a \quad (2.4a)$$

$$\text{and so } h_+(x) = -\frac{1}{4} \frac{x^2}{a} + \frac{1}{6} \frac{x^3}{a^2} \quad 0 < x < a$$

[We could add a constant to $h_+(x)$ but then we would have to subtract the same constant from $h_-(x)$, leaving y unchanged]

(2.3) then tells us that

$$\boxed{h_+(u) = -\frac{1}{4} \frac{u^2}{a} + \frac{1}{6} \frac{|u|^3}{a^2}, \quad -a < u < a} \quad (2.5)$$

which together with the period $2a$ defines $h_+(u)$ for all u .

[Note that this means we have a different algebraic formula for $h_+(u)$ in each of the intervals $(2n-1)a < u < (2n+1)a$, $n = 0, \pm 1, \pm 2, \dots$. For example,

$$h_+(u) = -\frac{1}{4} \frac{(u-2a)^2}{a} + \frac{1}{6} \frac{|u-2a|^3}{a^2}, \quad a < u < 3a$$

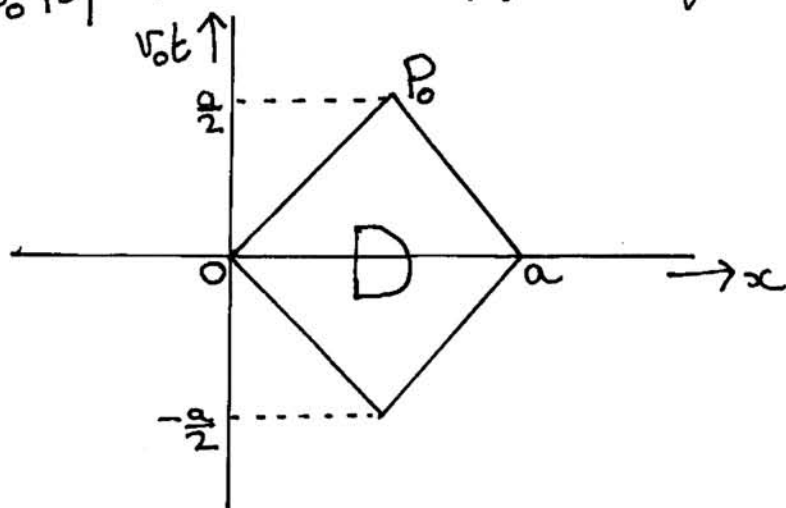
c) Using (2.1) and (2.3), solution for y is

$$y(t, x) = h_+(x - v_0 t) - h_+(x + v_0 t) \quad (2.6)$$

Provided $0 < x + v_0 t < a$ and $0 < x - v_0 t < a$
 we can use (2.5) (without the mod sign i.e. with $|u|^3 = u^3$)
 for h_+ , which gives

$$y(t, x) = v_0 t \frac{x}{a} \left(1 - \frac{x}{a}\right) - \frac{v_0^3 t^3}{3a^3}, \quad (x, t) \in D \quad (2.7)$$

D is the domain $-v_0 t < x < a - v_0 t$ and $v_0 t < x < a + v_0 t$
 i.e. $v_0 |t| < x < a - v_0 |t|$, which requires $v_0 |t| < \frac{1}{2}a$



[Note that D is the domain where y is determined by the
 initial values in $0 < x < a$, without using the boundary
 conditions at $x = 0$ and $x = a$]

d) From (2.6)

$$\frac{\partial y}{\partial t} = -v_0 h'_+(x - v_0 t) - v_0 h'_+(x + v_0 t)$$

At $t = t_0 = \frac{1}{2} \frac{a}{v_0}$ and $x = \frac{a}{2}$ (P_0 in the figure)

$$\frac{\partial y}{\partial t} = -v_0 \{ h'_+(0) + h'_+(a) \}$$

$$= 0 \quad \text{from (2.4a)}$$

From (2.6), at P_0 $y = h_+(0) - h_+(a)$

So from (2.5), $y = \frac{1}{12} a$

4.5 Closed string motion.

(a) If $x \sim x + 2\pi R$, then we must require that

$$y(x + 2\pi R, t) = y(x, t) . \quad (1)$$

In terms of the $h_{\pm}$ functions, this implies that

$$h_+(x - v_0 t + 2\pi R) + h_-(x + v_0 t + 2\pi R) = h_+(x - v_0 t) + h_-(x + v_0 t) , \quad (1')$$

Letting

$$u = x - v_0 t , \quad \text{and} \quad v = x + v_0 t \quad (1'')$$

equation (1') becomes

$$h_+(u + 2\pi R) + h_-(v + 2\pi R) = h_+(u) + h_-(v) , \quad (2)$$

Rearranging, Eq. (2) implies that

$$h_+(u + 2\pi R) - h_+(u) = - [h_-(v + 2\pi R) - h_-(v)] , \quad (3)$$

where the left-hand side depends only on u , while the right-hand side depends only on v . Although u and v both depend on x , they can nonetheless be treated as two independent variables, with x and t determined by

$$x = \frac{u + v}{2} , \quad \text{and} \quad t = \frac{v - u}{2v_0} . \quad (4)$$

If we differentiate Eq. (3) with respect to u , we find

$$h'_+(u + 2\pi R) - h'_+(u) = 0 , \quad (5)$$

where the prime denotes the derivative of the function with respect to its argument. This implies that $h'_+(u)$ is a periodic function of u , with period $2\pi R$. Similarly, by differentiating Eq. (3) with respect to v , we find that $h'_-(v)$ is a periodic function of v , with period $2\pi R$:

$$h'_-(v + 2\pi R) - h'_-(v) = 0 . \quad (5')$$

(b) Integrating Eq. (5), we can write

$$h_+(u + 2\pi R) - h_+(u) = \alpha(2\pi R) , \quad (6)$$

where α is a constant that was introduced as the average slope of $h_+(u)$. We can then construct a function $f(u)$ with zero average slope by defining

$$f(u) \equiv h_+(u) - \alpha u . \quad (7)$$

One can then see that the function f is in fact periodic with period $2\pi R$. Indeed,

$$f(u + 2\pi R) - f(u) = h_+(u + 2\pi R) - \alpha(u + 2\pi R) - [h_+(u) - \alpha u]$$

and therefore

$$f(u + 2\pi R) - f(u) = h_+(u + 2\pi R) - h_+(u) - \alpha(2\pi R) = 0. \quad (8)$$

Eq. (7) can be rewritten as

$$h_+(u) = \alpha u + f(u). \quad (9)$$

Similarly, the periodicity of $h'_-(v)$ allows us to write

$$h_-(v + 2\pi R) - h_-(v) = \beta(2\pi R), \quad (10)$$

and completely analogous steps allow us to obtain the analog of (9), which is

$$h_-(v) = \beta v + g(v), \quad (11)$$

where $g(v)$ is periodic with period $2\pi R$. From Eqs. (3), (6), and (10), one sees that

$$\alpha(2\pi R) = -\beta(2\pi R), \quad \text{or} \quad \boxed{\alpha + \beta = 0}. \quad (12)$$

(c) Taking the mass per length of the string as μ_0 , the total momentum in the y direction can be written as

$$\begin{aligned} p_y &= \mu_0 \int_0^{2\pi R} dx \frac{\partial y}{\partial t} \\ &= \mu_0 \int_0^{2\pi R} dx \left\{ \frac{dh_+}{du} \frac{\partial u}{\partial t} + \frac{dh_-}{dv} \frac{\partial v}{\partial t} \right\} \\ &= -\mu_0 v_0 \int_0^{2\pi R} dx \left\{ h'_+(x - v_0 t) - h'_-(x + v_0 t) \right\} \end{aligned}$$

Since the $h'_\pm$ functions are each periodic, and the integration is over one full period, we can reparameterize the region of integration as

$$\begin{aligned} p_y &= -\mu_0 v_0 \int_0^{2\pi R} du \left\{ h'_+(u) - h'_-(u) \right\} \\ &= -\mu_0 v_0 \int_0^{2\pi R} du \left\{ \alpha + f'(u) - \beta - g'(u) \right\} \\ &= 2\pi R \mu_0 v_0 (\beta - \alpha) - \mu_0 v_0 \left\{ [f(u + 2\pi R) - f(u)] - [g(u + 2\pi R) - g(u)] \right\}. \end{aligned}$$

Since f and g are periodic, the expression in curly brackets vanishes and finally

$$\boxed{p_y = 2\pi R \mu_0 v_0 (\beta - \alpha) = -4\pi R \mu_0 v_0 \alpha,}$$

which is manifestly independent of time, and is therefore conserved.

4.6 Stationary action: minima and saddles.

We have an action

$$S[x] = \int_0^{t_f} dt \frac{1}{2} m (\dot{x}^2 - \bar{\omega}^2 x^2)$$

(a) If $\bar{x}$ is a classical solution, it follows by Hamilton's principle that the variation ΔS of the action, defined as

$$\Delta S[\delta x] \equiv S[\bar{x} + \delta x] - S[\bar{x}]$$

cannot contain a term linear in δx . Since ΔS also vanishes for δx equal to zero, ΔS only contains terms quadratic, or possibly higher order in δx . But since the action S is quadratic in the dynamical variable we have that ΔS is in fact equal to $S[x]$ evaluated for $x \rightarrow \delta x$:

$$\Delta S[\delta x] = \int_0^{t_f} dt \frac{1}{2} m \left(\left(\frac{d\delta x}{dt} \right)^2 - \bar{\omega}^2 \delta x^2 \right). \quad (1)$$

(b) We want to evaluate $\Delta S[\delta_n x]$ for $\delta_n x = \sin \omega_n t$ with $\omega_n = \pi n / t_f$. Using (1) we find

$$\begin{aligned} \Delta S[\delta_n x] &= \int_0^{t_f} dt \frac{1}{2} m \left(\omega_n^2 \cos^2 \omega_n t - \bar{\omega}^2 \sin^2 \omega_n t \right) \\ &= \int_0^{t_f} dt \frac{1}{4} m \left(\omega_n^2 (1 + \cos 2\omega_n t) - \bar{\omega}^2 (1 - \cos 2\omega_n t) \right) \\ &= \frac{1}{4} m (\omega_n^2 - \bar{\omega}^2) t_f, \end{aligned}$$

since the terms with $\cos 2\omega_n t$ integrate to zero over the interval $[0, t_f]$. We thus obtained

$$\boxed{\Delta S[\delta_n x] = \frac{1}{4} m (\omega_n^2 - \bar{\omega}^2) t_f.} \quad (2)$$

For a variation that is a linear superposition

$$\delta x = \sum_{n=1}^{\infty} b_n \delta_n x = b_1 \delta_1 x + b_2 \delta_2 x + \dots$$

the calculation of $\Delta S[\delta x]$ contains cross products. Consider the cross product that involves $b_i \delta_i x = b_i \sin \omega_i t$ and $b_j \delta_j x = b_j \sin \omega_j t$. This cross product in ΔS receives contributions from the integral

$$\int_0^{t_f} dt b_i b_j \left(\frac{d}{dt} \sin \omega_i t \right) \left(\frac{d}{dt} \sin \omega_j t \right) = b_i b_j \omega_i \omega_j \int_0^{t_f} dt \cos \omega_i t \cos \omega_j t$$

that arises from the $(\frac{d\delta x}{dt})^2$ term in (1) and the integral

$$b_i b_j \int_0^{t_f} dt \sin \omega_i t \sin \omega_j t$$

that arises from the $(\delta x)^2$ term in (1). But we now claim that

$$\int_0^{t_f} dt \cos \omega_i t \cos \omega_j t = \int_0^{t_f} dt \sin \omega_i t \sin \omega_j t = 0, \quad \text{for } i \neq j. \quad (3)$$

This is a consequence of the trigonometric identities

$$\begin{aligned} \cos A \cos B &= \frac{1}{2} (\cos(A+B) + \cos(A-B)) \\ \sin A \sin B &= \frac{1}{2} (\cos(A-B) - \cos(A+B)). \end{aligned}$$

For example

$$\begin{aligned} \int_0^{t_f} dt \cos \omega_i t \cos \omega_j t &= \frac{1}{2} \int_0^{t_f} dt \cos(\omega_i + \omega_j)t + \frac{1}{2} \int_0^{t_f} dt \cos(\omega_i - \omega_j)t \\ &= \frac{1}{2} \int_0^{t_f} dt \cos \omega_{i+j}t + \frac{1}{2} \int_0^{t_f} dt \cos \omega_{|i-j|}t \\ &= \frac{1}{2\omega_{i+j}} \sin \omega_{i+j}t \Big|_0^{t_f} + \frac{1}{2\omega_{|i-j|}} \sin \omega_{|i-j|}t \Big|_0^{t_f} \end{aligned}$$

The sin functions that appear to the right, for $i \neq j$, are in fact our basis functions that vanish at the endpoints of the time interval. Thus the integral vanishes. The other integral in (3) vanishes for completely analogous reasons. Since all cross products vanish, we indeed have that

$$\Delta S \left[\sum_{n=1}^{\infty} b_n \delta_n x \right] = \sum_{n=1}^{\infty} \Delta S[b_n \delta_n x], \quad (4)$$

as we wanted to prove.

(c) It is useful to recast (4) in a more explicit way. First note that because of (1) we also have that

$$\Delta S[b_n \delta_n x] = b_n^2 \Delta S[\delta_n x]. \quad (5)$$

Equations (5) and (2) imply that (4) gives

$$\Delta S \left[\sum_{n=1}^{\infty} b_n \delta_n x \right] = \sum_{n=1}^{\infty} b_n^2 \frac{1}{4} m (\omega_n^2 - \bar{\omega}^2) t_f = \frac{1}{4} m t_f \sum_{n=1}^{\infty} b_n^2 (\omega_n^2 - \bar{\omega}^2) \quad (6)$$

We write this as

$$\boxed{\Delta S \left[\sum_{n=1}^{\infty} b_n \delta_n x \right] = \frac{1}{4} m t_f \sum_{n=1}^{\infty} b_n^2 (\omega_n^2 - \bar{\omega}^2) .} \quad (7)$$

We see that ΔS is positive for all values of the b_n coefficients if $\omega_n > \bar{\omega}$ for all values of n . Since $\omega_n \sim n$, this just requires

$$\omega_1 > \bar{\omega} \rightarrow \frac{\pi}{t_f} > \bar{\omega} \rightarrow t_f < \frac{\pi}{\bar{\omega}}$$

Recalling that $\bar{\omega} = 2\pi/\bar{T}$, with $\bar{T}$ the period, we see that we have a *minimum* of the action for solutions that extend for times t_f with $t_f < \bar{T}/2$, namely, less than half a period.

To have only δ_1 lead to a variation that lowers the action we must have

$$\omega_1 < \bar{\omega} \quad \text{and} \quad \omega_2 > \bar{\omega}$$

This gives

$$\frac{\pi}{t_f} < \bar{\omega} \quad \text{and} \quad \frac{2\pi}{t_f} > \bar{\omega}$$

or equivalently

$$\frac{\pi}{\bar{\omega}} < t_f < \frac{2\pi}{\bar{\omega}}$$

This is the range $\frac{\bar{T}}{2} < t_f < \bar{T}$. Each time t_f increases by half a period, an additional variation decreases the action. Thus, in general, a solution is a stationary point that corresponds to a saddle with a finite number of independent variations that lower the action and an infinite number of independent variations that increase the action.

4.7 Variational problem for strings

a) Equation (4.20) is
$$\left. \begin{aligned} T_0 \frac{d^2 y}{dx^2} + \omega^2 \mu(x) y &= 0 \\ y(0) = y(a) &= 0 \end{aligned} \right\} \quad (3.1)$$

which we assume determines a complete set of eigenvalues ω_n^2 and eigenfunctions $\psi_n(x)$, $n = 0, 1, 2, \dots$ in which any function $y(x)$ may be expanded as $y(x) = \sum_{n=0}^{\infty} C_n \psi_n(x)$.

We want to define an inner product such that $(\psi_i, \psi_j) = 0$, $i \neq j$.
[Then we can mimic the standard Fourier sine series expansion to get the coefficients $C_n = (\psi_n, y) / (\psi_n, \psi_n)$]

From
$$\frac{d^2 \psi_i^*}{dx^2} + \omega_i^2 \frac{\mu(x)}{T_0} \psi_i^* = 0$$

$$\frac{d^2 \psi_j}{dx^2} + \omega_j^2 \frac{\mu(x)}{T_0} \psi_j = 0$$

we get
$$\int_0^a \left(\psi_j \frac{d^2 \psi_i^*}{dx^2} - \psi_i^* \frac{d^2 \psi_j}{dx^2} \right) dx + \frac{(\omega_i^2 - \omega_j^2)}{T_0} \int_0^a \mu(x) \psi_i^* \psi_j dx = 0$$

[* means complex conjugate. We do this to make the inner product 'Hermitian' i.e. $(\psi_i, \psi_j) = (\psi_j, \psi_i)^*$ but in fact all our functions will be real so you can ignore the *s]

The first integrand is
$$\frac{d}{dx} \left(\psi_j \frac{d\psi_i^*}{dx} - \psi_i^* \frac{d\psi_j}{dx} \right)$$

Since ψ_i and ψ_j are both zero at $x=0$ and $x=a$, the first integral is zero, and so

$$\text{if } \omega_i^2 \neq \omega_j^2, \quad \int_0^a \mu(x) \psi_i^* \psi_j dx = 0 \quad (3.2)$$

This suggests that we define

$$(\phi, \psi) = \int_0^a \mu(x) \phi^*(x) \psi(x) dx \quad (3.3)$$

To follow the analogy with quantum mechanics, note that

$$\text{from (3.1)} \quad \omega_0^2 = \frac{-T_0 \int \psi_0^* \frac{d^2 \psi_0}{dx^2} dx}{(\psi_0, \psi_0)} \quad (3.4)$$

$$\text{Define an operator } H \text{ by } (H\psi)(x) = -\frac{T_0}{\mu(x)} \frac{d^2 \psi}{dx^2} \quad (3.5)$$

$$\text{Then (3.4) becomes } \omega_0^2 = \frac{(\psi_0, H\psi_0)}{(\psi_0, \psi_0)} \quad (3.6)$$

[Note that with this definition and (3.3), H is hermitian,

$$\text{i.e. } (\phi, H\psi) = (\psi, H\phi)^* \text{ — the } \mu(x) \text{ factors cancel}]$$

Provided $\psi(0) = \psi(a) = 0$, we can integrate by parts to get

$$(\psi, H\psi) = T_0 \int \left| \frac{d\psi}{dx} \right|^2 dx \quad (3.7)$$

[which is twice the elastic energy of the string with profile $\psi(x)$]

$[\omega^2(\psi, \psi)$ is twice the maximum kinetic energy with profile $\psi(x) \sin \omega t]$

Since $H\psi_i = \omega_i^2 \psi_i$, it follows from (3.2) that if $\omega_j^2 \neq \omega_i^2$, $(\psi_j, H\psi_i) = 0$ (3.8)

Fix the normalization of ψ_i so that $(\psi_i, \psi_i) = 1$ and hence

$$\left. \begin{aligned} (\psi_i, \psi_j) &= \delta_{ij} \\ \text{and} \quad (\psi_i, H\psi_j) &= \omega_i^2 \delta_{ij} \end{aligned} \right\} \quad (3.9)$$

Now choose any function $\psi(x)$ with $\psi(0) = \psi(a) = 0$ and expand it as $\psi(x) = \sum_0^\infty C_n \psi_n(x)$

$$\text{From (3.9),} \quad \frac{(\psi, H\psi)}{(\psi, \psi)} = \frac{\sum_0^\infty |C_n|^2 \omega_n^2}{\sum_0^\infty |C_n|^2} \quad (3.10)$$

It follows that $\frac{(\psi, H\psi)}{(\psi, \psi)} \geq \omega_0^2$

and only equals ω_0^2 when $C_1 = C_2 = \dots = 0$ i.e. $\psi = \psi_0$.

Using (3.7), we get

$$\omega_0^2 \leq \frac{T_0 \int_0^a \left| \frac{d\psi}{dx} \right|^2 dx}{\int_0^a \mu(x) |\psi(x)|^2 dx} \quad (3.11)$$

where $\psi(x)$ is any function vanishing at the ends.

Our variational procedure^{*} is to choose a family of functions $\psi(x, C_1, C_2, \dots, C_r)$ depending on r parameters C_i , calculate $\omega^2(C_1, \dots, C_r) = T_0 \int_0^a \left| \frac{d\psi}{dx} \right|^2 dx / \int_0^a \mu(x) |\psi|^2 dx$ and minimize with respect to $C_1, \dots, C_r$.

The simplest version is $r=0$! i.e. guess a function ψ and calculate ω^2 from (3.11)

[* If we start with the right hand side of (3.11) and use the calculus of variations to minimize it, we get (3.1). Try it.]

b) $\mu(x) = \mu_0 \frac{x}{a}$. Try $\psi = \frac{x}{a} (1 - \frac{x}{a})$ and let $u = \frac{x}{a}$

$$\text{Then } \omega_0^2 \leq \frac{T_0}{\mu_0 a^2} \int_0^1 (1-2u)^2 du / \int_0^1 u u^2 (1-u)^2 du$$

$$= \frac{T_0}{\mu_0 a^2} \frac{1 - 4 \cdot \frac{1}{2} + 4 \cdot \frac{1}{3}}{\frac{1}{4} - 2 \cdot \frac{1}{5} + \frac{1}{6}} = 20 \frac{T_0}{\mu_0 a^2}$$

$$\left[\psi = \sin \frac{\pi x}{a} \text{ gives } \omega_0^2 \leq 2\pi^2 \frac{T_0}{\mu_0 a^2} = 19.74 \frac{T_0}{\mu_0 a^2} \right]$$

4.8 Deriving Euler-Lagrange equations.

(a) Under a variation $\delta q(t)$ of the coordinate, the variation $\delta \dot{q}(t)$ of the velocity is $\frac{d\delta q}{dt}$. Since the Lagrangian L depends on q and $\dot{q}$, the full variation is

$$\delta S = \int dt \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d\delta q}{dt} \right\}.$$

The second term in the brackets is rewritten in terms of a total time derivative and a term proportional to δq :

$$\delta S = \int dt \left\{ \frac{\partial L}{\partial q} \delta q + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right\}$$

The total time derivative does not contribute when we set the variations to vanish at the initial and final times (we always assume this). The variation δS is therefore

$$\delta S = \int dt \delta q(t) \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right\}.$$

If this variation is to vanish for all $\delta q(t)$ we must have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0. \quad (1)$$

This is the Euler-Lagrange equation.

(b) Under a variation $\delta \phi(x)$ of the field, the variation $\delta(\partial_\mu \phi)$ of the field derivative is equal to $\partial_\mu(\delta \phi)$. Since the Lagrangian density $\mathcal{L}$ depends on ϕ and $\partial_\mu \phi$, the full variation is

$$\begin{aligned} \delta S &= \int d^D x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta \phi) \right\} \\ &= \int d^D x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta \phi \right\}. \end{aligned}$$

The total derivatives (middle term) do not contribute if the variations vanish at infinity or, more precisely, if $\delta \phi(x^0, x^1, x^2, \dots, x^d)$ vanishes when any of its arguments is equal to plus or minus infinity. Consider, for example, the term with $\mu = 1$:

$$\int dx^0 dx^2 \dots dx^d \frac{\partial}{\partial x^1} \left(\frac{\partial \mathcal{L}}{\partial(\partial_1 \phi)} \delta \phi \right) = \int dx^0 dx^2 \dots dx^d \left[\frac{\partial \mathcal{L}}{\partial(\partial_1 \phi)} \delta \phi \right]_{x^1=-\infty}^{x^1=\infty} = 0,$$

if $\delta \phi(x^0, x^1 = \pm\infty, x^2, \dots, x^d) = 0$. Back to the variation, we therefore have

$$\delta S = \int d^D x \delta \phi(x) \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right\} = 0 \quad \rightarrow \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0,$$

which is the Euler-Lagrange equation for the field.

5.1 Point particle equation of motion and reparametrizations

$$\frac{d^2 x^\mu}{ds^2} = 0 \quad \tau = f(s)$$

$$\frac{dx^\mu}{ds} = \frac{dx^\mu}{d\tau} \frac{d\tau}{ds} = \frac{dx^\mu}{d\tau} f'(s)$$

$$\frac{d^2 x^\mu}{ds^2} = \frac{d^2 x^\mu}{d\tau^2} [f'(s)]^2 + \frac{dx^\mu}{d\tau} f''(s)$$

so equation of motion is $\frac{d^2 x^\mu}{d\tau^2} = 0$ if and only if $f''(s) = 0$

i.e. $f(s) = As + B$ with A, B constant

i.e. allowed reparametrizations are a shift of origin and a rescaling.

5.2 Particle equation of motion with arbitrary parameterization.

We are told to vary $x^\mu(\tau) \rightarrow x^\mu(\tau) + \delta x^\mu(\tau)$ in the action

$$S = -mc \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau.$$

The variation gives

$$\delta S = -mc \int_{\tau_i}^{\tau_f} \frac{1}{2} \frac{-2\eta_{\mu\nu} \frac{d\delta x^\mu}{d\tau} \frac{dx^\nu}{d\tau}}{\sqrt{-\eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}}} d\tau = mc \int_{\tau_i}^{\tau_f} \frac{\frac{d\delta x^\mu}{d\tau} \frac{dx_\mu}{d\tau}}{\sqrt{-\eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}}} d\tau.$$

Integrating by parts and dropping total derivatives ($\delta x^\mu(\tau_i) = \delta x^\mu(\tau_f) = 0$) we find

$$\delta S = - \int_{\tau_i}^{\tau_f} d\tau \delta x^\mu(\tau) \frac{d}{d\tau} \left[\frac{mc \frac{dx_\mu}{d\tau}}{\sqrt{-\eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}}} \right].$$

The action is stationary if

$$\frac{d}{d\tau} \left[\frac{mc \frac{dx_\mu}{d\tau}}{\sqrt{-\eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}}} \right] = 0. \quad (1)$$

This is the equation of motion for the particle in “manifestly” reparameterization invariant form. Indeed, the object between the brackets is clearly reparameterization invariant:

$$\frac{mc \frac{dx_\mu}{d\tau}}{\sqrt{-\eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}}} = \frac{mc \frac{dx_\mu}{d\tau'}}{\sqrt{-\eta_{\alpha\beta} \frac{dx^\alpha}{d\tau'} \frac{dx^\beta}{d\tau'}}}.$$

(This follows immediately using the chain rule $\frac{dx}{d\tau} = \frac{d\tau'}{d\tau} \frac{dx}{d\tau'}$.) The derivative in front of the brackets (see (1)) does not spoil the reparameterization invariance since:

$$\frac{d}{d\tau}[\dots] = \frac{d\tau'}{d\tau} \frac{d}{d\tau'}[\dots] = 0 \quad \rightarrow \quad \frac{d}{d\tau'}[\dots] = 0.$$

When we choose τ equal to s

$$-\eta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = -\eta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \frac{(ds)^2}{(ds)^2} = 1.$$

Equation (1) then becomes the familiar

$$\frac{d}{ds} \left[mc \frac{dx_\mu}{ds} \right] = \frac{dp_\mu}{ds} = 0.$$

5.3 Current of a charged point particle

$$a) \quad j^0(\vec{x}, t) = qc \delta^d(\vec{x} - \vec{X}(t)) \quad (2.1)$$

$$\vec{j}(\vec{x}, t) = q \frac{d\vec{X}}{dt} \delta^d(\vec{x} - \vec{X}(t)) \quad (2.2)$$

where $\vec{X}(t)$ is the position in d -dimensional space of the particle at time t i.e. $\vec{X}(t) = \vec{x}^0(\tau)$ where $ct = x^0(\tau)$

$$b) \quad \text{Suppose } j^\mu(\vec{x}, t) = qc \int d\tau \frac{dx^\mu(\tau)}{d\tau} \delta^{d+1}(x - x(\tau)) \quad (2.3)$$

Consider the integral $\int d\tau \delta(x^0 - x^0(\tau)) F(\tau)$

The δ -function picks out the value of τ for which $x^0(\tau) = x^0$.

Change the integration variable to $X^0 = x^0(\tau)$.

$d\tau = dX^0 \frac{d\tau}{dX^0} = \frac{dX^0}{dx^0/d\tau}$ so the integral is $\frac{F(\tau)}{dx^0/d\tau}$

evaluated at the point where $x^0(\tau) = x^0$.

(Assuming that $\frac{dx^0}{d\tau}$ is always positive which is a sensible restriction to impose on our parametrization)

In (2.3), $F(\tau) = qc \delta^d(\vec{x} - \vec{x}(\tau)) \frac{dx^\mu}{d\tau}$, so we get

$$j^\mu(\vec{x}, t) = qc \delta^d(\vec{x} - \vec{x}(\tau)) \frac{dx^\mu}{d\tau} / \frac{dx^0}{d\tau} = qc \frac{dx^\mu}{dx^0} \delta^d(\vec{x} - \vec{x}(\tau))$$

evaluated where $x^0(\tau) = x^0$.

Since $\frac{dx^0}{dx^0} = 1$ and $\frac{d\vec{x}}{dx^0} = \frac{1}{c} \frac{d\vec{x}}{dt}$, this is exactly (2.1) and (2.2)

[Alternative way of saying this: (2.3) is clearly reparametrization invariant so choose $\tau = X^0$]

5.4 Hamiltonian for a non-relativistic charged particle

$$S = \int \frac{1}{2} m \dot{\mathbf{r}}^2 dt + \frac{q}{c} \int A_\mu \frac{dx^\mu}{dt} dt \quad \text{where } A^\mu = (\Phi, \vec{A})$$

$$a) \quad S = \int \frac{1}{2} m \dot{\mathbf{r}}^2 dt + \frac{q}{c} \int \vec{A} \cdot \vec{v} dt - q \int \Phi dt = \int L dt$$

$$\text{where } L = \frac{1}{2} m \dot{\mathbf{r}}^2 + \frac{q}{c} \vec{A} \cdot \vec{v} - q \Phi$$

$\left[\frac{1}{2} m \dot{\mathbf{r}}^2 - q \Phi \right]$ is the familiar kinetic energy minus potential energy
The new 'magnetic' term is linear in $\vec{v}$.

$$b) \quad \vec{p} = \frac{\partial L}{\partial \vec{v}} \quad (\text{definition of canonical momentum})$$

$$= m \vec{v} + \frac{q}{c} \vec{A}$$

$$c) \quad H = \vec{p} \cdot \vec{v} - L \quad (\text{definition of } H)$$

$$= m \dot{\mathbf{r}}^2 + \frac{q}{c} \vec{A} \cdot \vec{v} - \frac{1}{2} m \dot{\mathbf{r}}^2 - \frac{q}{c} \vec{A} \cdot \vec{v} + q \Phi$$

$$= \frac{1}{2} m \dot{\mathbf{r}}^2 + q \Phi$$

This looks as if it knows nothing about $\vec{A}$. However H must be expressed as a function of $\vec{p}$, not $\vec{v}$,
using $m \vec{v} = \vec{p} - \frac{q}{c} \vec{A}$, which gives

$$H = \frac{1}{2m} \left(\vec{p} - \frac{q}{c} \vec{A} \right)^2 + q \Phi$$

5.5 Equations of motion for a charged particle.

We are interested in the variation of the action

$$S = -mc \int_{\mathcal{P}} ds + \frac{q}{c} I, \quad I = \int_{\mathcal{P}} d\tau \, A_{\mu}(x(\tau)) \frac{dx^{\mu}}{d\tau}(\tau), \quad (1)$$

when we let $x^{\mu}(\tau) \rightarrow x^{\mu}(\tau) + \delta x^{\mu}(\tau)$. We note that

$$\delta A_{\mu}(x(\tau)) \equiv A_{\mu}(x(\tau) + \delta x(\tau)) - A_{\mu}(x(\tau)) = \frac{\partial A_{\mu}}{\partial x^{\nu}} \delta x^{\nu}(\tau),$$

where $\frac{\partial A_{\mu}}{\partial x^{\nu}}$ is calculated at $x = x(\tau)$. We therefore have

$$\delta I = \int_{\mathcal{P}} d\tau \, \frac{\partial A_{\mu}}{\partial x^{\nu}} \delta x^{\nu} \frac{dx^{\mu}}{d\tau} + \int_{\mathcal{P}} d\tau \, A_{\mu} \frac{d\delta x^{\mu}}{d\tau}.$$

We exchange $\mu \leftrightarrow \nu$ in the first term and rewrite the second using a total derivative:

$$\delta I = \int_{\mathcal{P}} d\tau \, \delta x^{\mu} \frac{\partial A_{\nu}}{\partial x^{\mu}} \frac{dx^{\nu}}{d\tau} + \int_{\mathcal{P}} d\tau \left[\frac{d}{d\tau} (A_{\mu} \delta x^{\mu}) - \delta x^{\mu} \frac{dA_{\mu}}{d\tau} \right].$$

We assume that δx vanishes at the ends of $\mathcal{P}$, so the total derivative (first term in brackets) vanishes. Using the chain rule for the second term in brackets we find

$$\delta I = \int_{\mathcal{P}} d\tau \, \delta x^{\mu} \frac{\partial A_{\nu}}{\partial x^{\mu}} \frac{dx^{\nu}}{d\tau} - \int_{\mathcal{P}} d\tau \, \delta x^{\mu} \frac{\partial A_{\mu}}{\partial x^{\nu}} \frac{dx^{\nu}}{d\tau}.$$

The two terms can now be combined to write

$$\delta I = \int_{\mathcal{P}} d\tau \, \delta x^{\mu} \left(\frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}} \right) \frac{dx^{\nu}}{d\tau} = \int_{\mathcal{P}} d\tau \, \delta x^{\mu} F_{\mu\nu} \frac{dx^{\nu}}{d\tau}.$$

This concludes our computation of the variation of I . The variation of the first term in (1) was calculated in the textbook (equation (5.29)). The total variation is therefore

$$\begin{aligned} \delta S &= - \int_{\mathcal{P}} d\tau \, \delta x^{\mu} \frac{dp_{\mu}}{d\tau} + \frac{q}{c} \int_{\mathcal{P}} d\tau \, \delta x^{\mu} F_{\mu\nu} \frac{dx^{\nu}}{d\tau} \\ &= \int_{\mathcal{P}} d\tau \, \delta x^{\mu} \left(-\frac{dp_{\mu}}{d\tau} + F_{\mu\nu} \frac{dx^{\nu}}{d\tau} \right). \end{aligned}$$

Setting the variation equal to zero, we find the expected equation of motion

$$\frac{dp_{\mu}}{d\tau} = \frac{q}{c} F_{\mu\nu} \frac{dx^{\nu}}{d\tau}.$$

5.6 Electromagnetic field dynamics with a charged point particle.

Write

$$S_{int} = \frac{q}{c} \int_{\mathcal{P}} d\tau A_\mu(x(\tau)) \frac{dx^\mu}{d\tau}(\tau).$$

Since

$$A_\mu(x(\tau)) = \int d^D x \delta^D(x - x(\tau)) A_\mu(x),$$

we have

$$S_{int} = \frac{1}{c^2} \int d^D x A_\mu(x) \left[qc \int_{\mathcal{P}} d\tau \delta^D(x - x(\tau)) \frac{dx^\mu}{d\tau} \right] = \frac{1}{c^2} \int d^D x A_\mu(x) j^\mu(x),$$

where $j^\mu(x)$ is the current of Problem 5.3. Under a variation $\delta A_\mu(x)$ of the gauge field

$$\delta S_{int} = \frac{1}{c} \int d^D x \delta A_\mu(x) \frac{j^\mu(x)}{c}.$$

Let now

$$S_{em} = -\frac{1}{4c} \int d^D x F_{\mu\nu} F^{\mu\nu}.$$

We then have

$$\begin{aligned} \delta S_{em} &= -\frac{1}{4c} \int d^D x 2 \cdot F^{\mu\nu} \delta F_{\mu\nu} = -\frac{1}{2c} \int d^D x \cdot F^{\mu\nu} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) \\ &= +\frac{1}{c} \int d^D x F^{\mu\nu} \frac{\partial \delta A_\mu}{\partial x^\nu} \quad (\text{since } F^{\mu\nu} = -F^{\nu\mu}) \\ &= -\frac{1}{c} \int d^D x \delta A_\mu \frac{\partial F^{\mu\nu}}{\partial x^\nu} \quad (\text{throwing away boundary terms}). \end{aligned}$$

Collecting variations we have

$$\delta S_{int} + \delta S_{em} = -\frac{1}{c} \int d^D x \delta A_\mu(x) \left(\frac{\partial F^{\mu\nu}}{\partial x^\nu} - \frac{j^\mu(x)}{c} \right).$$

Setting the total variation to vanish we find the field equations

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{j^\mu}{c}.$$

5.7 Point particle action in curved space.

Our starting point is the variation

$$\delta S = -mc \int \delta(ds). \quad (1)$$

With the path parameterized by an arbitrary τ we have

$$(ds)^2 = -g_{\mu\nu}(x(\tau)) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (d\tau)^2. \quad (2)$$

Under a variation $x(\tau) \rightarrow x(\tau) + \delta x(\tau)$ the variation of the metric is

$$\delta g_{\mu\nu}(x(\tau)) = g_{\mu\nu}(x + \delta x) - g_{\mu\nu}(x) = \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \delta x^\alpha(\tau),$$

so the variation of (2) gives

$$2 ds \delta(ds) = -\frac{\partial g_{\mu\nu}}{\partial x^\alpha} \delta x^\alpha \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (d\tau)^2 - 2g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d\delta x^\nu}{d\tau} (d\tau)^2.$$

Dividing by $2ds$ and cancelling one factor of $d\tau$, we find

$$\delta(ds) = -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \delta x^\alpha \frac{dx^\mu}{ds} \frac{dx^\nu}{d\tau} d\tau - g_{\mu\alpha} \frac{dx^\mu}{ds} \frac{d\delta x^\alpha}{d\tau} d\tau.$$

Back into (1), we have

$$\delta S = mc \int d\tau \left(\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \delta x^\alpha \frac{dx^\mu}{ds} \frac{dx^\nu}{d\tau} + g_{\mu\alpha} \frac{dx^\mu}{ds} \frac{d\delta x^\alpha}{d\tau} \right).$$

Integrating by parts and dropping total derivatives that vanish since $\delta x = 0$ at the endpoints of the world-line

$$\delta S = mc \int d\tau \delta x^\alpha \left(\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{d\tau} - \frac{d}{d\tau} \left[g_{\mu\alpha} \frac{dx^\mu}{ds} \right] \right).$$

The equation of motion simply requires the vanishing of the object inside parentheses. We thus have

$$\frac{d}{d\tau} \left[g_{\mu\alpha} \frac{dx^\mu}{ds} \right] = \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{d\tau}.$$

This is the geodesic equation. It is of course possible to choose the arbitrary parameter τ equal to s [Note: if we had used s as a parameter from the start, we might worry about the change in the range of s when the path changes.] We then have

$$\frac{d}{ds} \left[g_{\mu\alpha} \frac{dx^\mu}{ds} \right] = \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}.$$

(3)

The above form of the geodesic equation is the most useful for explicit computations. Nevertheless there is a familiar alternative version. Although this was not asked for in the problem, we discuss it below. Expanding out derivatives in (3) we get

$$g_{\mu\alpha} \frac{d^2 x^\mu}{ds^2} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} \frac{dx^\nu}{ds} \frac{dx^\mu}{ds} - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.$$

The last two terms in the left-hand side can be combined:

$$g_{\mu\alpha} \frac{d^2 x^\mu}{ds^2} + \frac{1}{2} \left(2 \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.$$

Since the expression in parenthesis multiplies $\frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$, which is symmetric in μ, ν , we rewrite it in a form which is also symmetric in μ, ν , to get

$$g_{\mu\alpha} \frac{d^2 x^\mu}{ds^2} + \frac{1}{2} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.$$

Multiplying the equation by $g^{\lambda\alpha}$ and recalling that $g^{\lambda\alpha} g_{\mu\alpha} = \delta_\mu^\lambda$, we have

$$\frac{d^2 x^\lambda}{ds^2} + \frac{1}{2} g^{\lambda\alpha} \left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.$$

The equation has taken the form

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma_{\mu\nu}^\lambda \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0,$$

with

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\alpha} \left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right).$$

6.1 The induced metric on a two-dimensional surface.

We have

$$\vec{x} = (x, y, h(x, y)),$$

and we are asked to use

$$\xi^1 = x, \quad \xi^2 = y.$$

We then have

$$\begin{aligned} \frac{\partial \vec{x}}{\partial \xi^1} &= \frac{\partial \vec{x}}{\partial x} = \left(1, 0, \frac{\partial h}{\partial x} \right), \\ \frac{\partial \vec{x}}{\partial \xi^2} &= \frac{\partial \vec{x}}{\partial y} = \left(0, 1, \frac{\partial h}{\partial y} \right). \end{aligned}$$

(a) We can now calculate using (6.14)

$$\begin{aligned} g_{11} &= \frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^1} = 1 + \left(\frac{\partial h}{\partial x} \right)^2, \\ g_{22} &= \frac{\partial \vec{x}}{\partial \xi^2} \cdot \frac{\partial \vec{x}}{\partial \xi^2} = 1 + \left(\frac{\partial h}{\partial y} \right)^2, \\ g_{12} = g_{21} &= \frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2} = \frac{\partial h}{\partial x} \frac{\partial h}{\partial y}. \end{aligned}$$

This information is summarized by

$$g_{ij} = \delta_{ij} + \frac{\partial h}{\partial \xi^i} \frac{\partial h}{\partial \xi^j}.$$

(b) We have

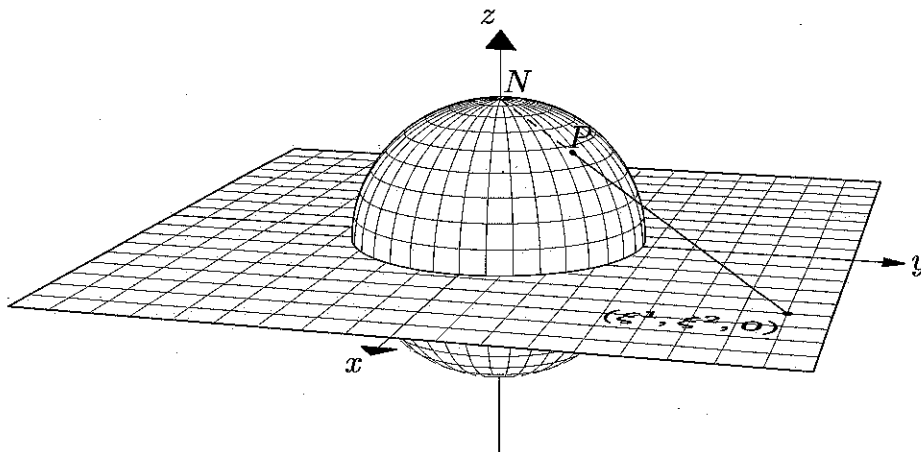
$$\begin{aligned} g &\equiv \det g_{ij} = g_{11}g_{22} - g_{12}^2 \\ &= \left[1 + \left(\frac{\partial h}{\partial x} \right)^2 \right] \left[1 + \left(\frac{\partial h}{\partial y} \right)^2 \right] - \left(\frac{\partial h}{\partial x} \right)^2 \left(\frac{\partial h}{\partial y} \right)^2 \\ &= 1 + \left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial y} \right)^2 \end{aligned}$$

The area integral is therefore:

$$A = \int d\xi^1 d\xi^2 \sqrt{g} = \int dx dy \sqrt{1 + \left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial y} \right)^2}.$$

Problem 6.2 Metric on S^2 from stereographic parameterization

- (a) A diagram of the construction is shown below. To parameterize the full plane except for the north pole, ξ^1 and ξ^2 must vary over the entire plane. The north pole is approached in the limit as $\sqrt{(\xi^1)^2 + (\xi^2)^2} \rightarrow \infty$, in any direction.

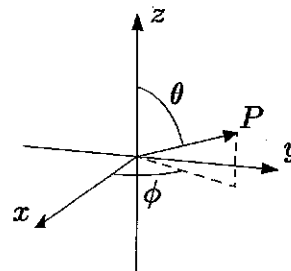


- (b) To calculate the cartesian coordinates $x^i(\xi^1, \xi^2)$ of the point P , one route is to first express the point in terms of spherical polar coordinates, as shown at the right. The spherical polar coordinates (r, θ, ϕ) are related to the cartesian coordinates by

$$x = r \sin \theta \cos \phi ,$$

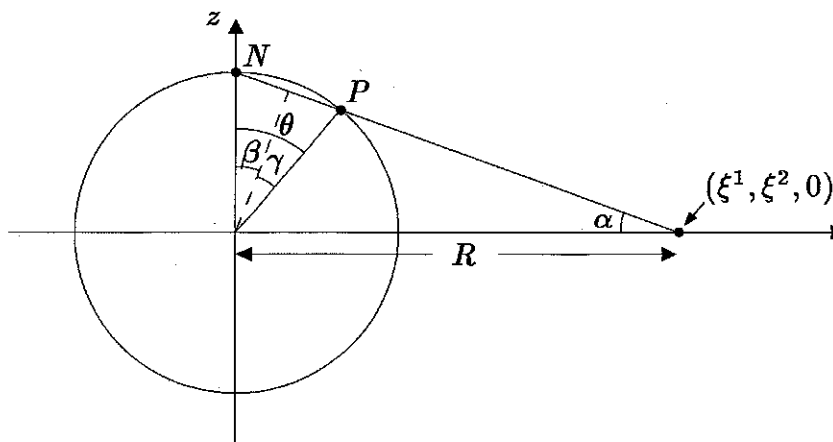
$$y = r \sin \theta \sin \phi ,$$

$$z = r \cos \theta .$$



(1)

To find the θ -coordinate for the point P , it is useful to draw a diagram showing the plane containing the z axis and P :



Here

$$R = \sqrt{(\xi^1)^2 + (\xi^2)^2} . \quad (2)$$

On the diagram, one can see that $\beta = \alpha$, since the two lines that make the angle β are perpendicular to the two lines that make the angle α , respectively. Then one notes that $\gamma = \beta$, since together they make an isosceles triangle that is bisected by the broken line. Finally, $\theta = \beta + \gamma = 2\alpha$, so

$$\begin{aligned} \sin \theta &= \sin 2\alpha = 2 \sin \alpha \cos \alpha = 2 \frac{1}{\sqrt{R^2 + 1}} \frac{R}{\sqrt{R^2 + 1}} = \frac{2R}{R^2 + 1} \\ \cos \theta &= \cos 2\alpha = 1 - 2 \sin^2 \alpha = 1 - \frac{2}{R^2 + 1} = \frac{R^2 - 1}{R^2 + 1} . \end{aligned} \quad (3)$$

The ϕ -coordinate for P is the same as the ϕ -coordinate for the point $(\xi^1, \xi^2, 0)$, so

$$\sin \phi = \frac{\xi^2}{R} , \quad \cos \phi = \frac{\xi^1}{R} . \quad (4)$$

By combining Eqs. (1), (3), and (4), and remembering that the sphere has unit radius, one sees that

$$\begin{aligned} x &= \frac{2\xi^1}{R^2 + 1} \\ y &= \frac{2\xi^2}{R^2 + 1} \\ z &= \frac{R^2 - 1}{R^2 + 1} . \end{aligned} \quad (5)$$

It is worth checking that these coordinates describe a unit sphere:

$$\begin{aligned} x^2 + y^2 + z^2 &= \frac{1}{(R^2 + 1)^2} [4(\xi^1)^2 + 4(\xi^2)^2 + (R^2 - 1)^2] \\ &= \frac{1}{(R^2 + 1)^2} [4R^2 + (R^2 - 1)^2] \\ &= \frac{1}{(R^2 + 1)^2} [R^4 + 2R^2 + 1] = 1 . \end{aligned} \quad (6)$$

They do!

(c) The metric is given by Eq. (6.14) of the text:

$$g_{ij}(\xi) \equiv \frac{\partial \vec{x}}{\partial \xi^i} \cdot \frac{\partial \vec{x}}{\partial \xi^j} . \quad (7)$$

As the problem suggested, this might be a good place to invoke a symbolic manipulator, and indeed the program I use was able to get the answer in a flash. Nonetheless, here I will show how to get the answer by hand, even if it is a little tedious. Using i, j , and k to denote indices that take the values 1 or 2, one can write

$$\frac{\partial R}{\partial \xi^i} = \frac{\xi^i}{R} . \quad (8)$$

The first two of Eqs. (5) can be written as

$$x^i = \frac{2\xi^i}{R^2 + 1} , \quad (9)$$

from which we find

$$\begin{aligned} \frac{\partial x^i}{\partial \xi^j} &= \frac{2\delta_j^i}{R^2 + 1} - \frac{2\xi^i}{(R^2 + 1)^2} 2R \frac{\partial R}{\partial \xi^j} \\ &= \frac{2\delta_j^i}{R^2 + 1} - \frac{4\xi^i \xi^j}{(R^2 + 1)^2} \\ &= \frac{2}{(R^2 + 1)^2} [\delta_j^i (R^2 + 1) - 2\xi^i \xi^j] . \end{aligned} \quad (10)$$

Then

$$\begin{aligned} \frac{\partial z}{\partial \xi^j} &= \frac{\partial z}{\partial R} \frac{\partial R}{\partial \xi^j} \\ &= \left[\frac{2R}{R^2 + 1} - \frac{R^2 - 1}{(R^2 + 1)^2} 2R \right] \frac{\xi^j}{R} \\ &= \frac{4\xi^j}{(R^2 + 1)^2} . \end{aligned} \quad (11)$$

Using Eq. (7), one sees that

$$\begin{aligned} g_{ij}(\xi) &= \frac{\partial x^k}{\partial \xi^i} \frac{\partial x^k}{\partial \xi^j} + \frac{\partial z}{\partial \xi^i} \frac{\partial z}{\partial \xi^j} \\ &= \frac{4}{(R^2 + 1)^4} [\delta_i^k (R^2 + 1) - 2\xi^k \xi^i] [\delta_j^k (R^2 + 1) - 2\xi^k \xi^j] + \frac{16\xi^i \xi^j}{(R^2 + 1)^4} \\ &= \frac{4}{(R^2 + 1)^4} [\delta_{ij} (R^2 + 1)^2 - 4\xi^i \xi^j (R^2 + 1) + 4\xi^i \xi^j R^2] + \frac{16\xi^i \xi^j}{(R^2 + 1)^4} \\ &= \frac{4}{(R^2 + 1)^4} [\delta_{ij} (R^2 + 1)^2 - 4\xi^i \xi^j] + \frac{16\xi^i \xi^j}{(R^2 + 1)^4} \\ &= \frac{4}{(R^2 + 1)^2} \delta_{ij} , \end{aligned} \quad (12)$$

so finally

$$g_{ij}(\xi) = \frac{4}{[(\xi^1)^2 + (\xi^2)^2 + 1]^2} \delta_{ij} . \quad (13)$$

This metric has the interesting property that it has the form $f(\xi)\delta_{ij}$, a function of position times the flat-space metric. This means that if one were to calculate an angle between two vectors $\vec{v}_1$ and $\vec{v}_2$, given by

$$\theta = \cos^{-1} \left[\frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| |\vec{v}_2|} \right] , \quad (14)$$

one would get the same answer using the metric of Eq. (13) as one would get using the flat-space metric, since the factors of $f(\xi)$ will cancel between the numerator and denominator. Such angle-preserving metrics are called "conformal to flat."

(d) We can now check our result by computing the area of the sphere, using

$$A = \int d\xi^1 d\xi^2 \sqrt{g} , \quad (15)$$

where

$$g = \det g_{ij} = \det \begin{pmatrix} \frac{4}{(R^2+1)^2} & 0 \\ 0 & \frac{4}{(R^2+1)^2} \end{pmatrix} = \left[\frac{4}{(R^2+1)^2} \right]^2 . \quad (16)$$

Using polar coordinates (R, θ) to describe the (ξ^1, ξ^2) plane,

$$A = \int_0^\infty R dR \int_0^{2\pi} d\theta \frac{4}{(R^2+1)^2} . \quad (17)$$

The integral over θ gives 2π , and the integral over R can be done by substituting $u \equiv R^2 + 1$, $du = 2R dR$:

$$A = 4\pi \int_1^\infty \frac{du}{u^2} = 4\pi , \quad (18)$$

which is, of course, the right answer.

(Solution by Alan Guth)

6.3 Schwarz inequality in $\mathbb{R}^{1,1}$.

(a) We are given a timelike vector t' and a spacelike vector s' . We can choose to keep either one of them as part of our canonical basis, up to normalization. I will choose to keep t' , so the normalized vector will be

$$t = \frac{t'}{\sqrt{-t'^2}} ,$$

so $t^2 = -1$. Once we have decided to use t as defined above, we cannot in general use s' , since there is no reason to believe that it is orthogonal to t . But we can form a vector

$$\tilde{s} = s' + \alpha t ,$$

where we choose α to make $\tilde{s}$ orthogonal to t :

$$\tilde{s} \cdot t = s' \cdot t + \alpha t^2 = s' \cdot t - \alpha = 0 ,$$

so

$$\alpha = s' \cdot t , \quad \text{and then} \quad \tilde{s} = s' + (s' \cdot t) t .$$

Provided that $\tilde{s}^2 > 0$, we can normalize $\tilde{s}$ and we are done. However, we must check that $\tilde{s}^2 > 0$, to make sure that we have achieved our goal:

$$\tilde{s}^2 = \tilde{s} \cdot \tilde{s} = \tilde{s} \cdot [s' + (s' \cdot t) t] .$$

By our construction we know that $\tilde{s} \cdot t = 0$, so the last step can be further simplified to give

$$\tilde{s}^2 = \tilde{s} \cdot s' = s'^2 + (s' \cdot t)^2 ,$$

which is positive definite. (While $s' \cdot t$ can have either sign, $(s' \cdot t)^2 \geq 0$ always.) So we can choose

$$s = \frac{\tilde{s}}{\sqrt{\tilde{s}^2}} .$$

If we wish to obtain an explicit expression for s directly in terms of s' and t' , straightforward substitution yields

$$s = \frac{(-t'^2)s' + (s' \cdot t') t'}{\sqrt{(-t'^2)^2 s'^2 + (-t'^2)(s' \cdot t')^2}} .$$

(b) We can expand the two arbitrary vectors in the basis vectors, writing

$$v_1 = at + bs , \quad v_2 = et + fs ,$$

where a , b , e , and f are real numbers. Note that I am trying to simplify the appearance of the algebra by avoiding subscripts and superscripts. We can then define

$$X \equiv (v_1 \cdot v_2)^2 - v_1^2 v_2^2 ,$$

so our goal is to prove that $X \geq 0$. Expanding,

$$\begin{aligned} X &= (bf - ae)^2 - (b^2 - a^2)(f^2 - e^2) \\ &= b^2 f^2 + a^2 e^2 - 2abef - (b^2 f^2 + a^2 e^2 - a^2 f^2 - e^2 b^2) \\ &= a^2 f^2 + e^2 b^2 - 2abef \\ &= (af - eb)^2 \geq 0 . \end{aligned}$$

Note that equality holds only when $X = 0$, which implies that

$$af - eb = 0 ,$$

which in turn implies that

$$\frac{a}{b} = \frac{e}{f} ,$$

which implies that v_1 and v_2 are parallel.

Alternatively, one could proceed similarly to the derivation used in the textbook for Eq. (6.37). This method begins by defining the vector

$$v^\mu(\lambda) \equiv v_1^\mu + \lambda v_2^\mu .$$

Then

$$v^2(\lambda) = \lambda^2 v_2^2 + 2\lambda v_1 \cdot v_2 + v_1^2 ,$$

and the equation $v^2(\lambda) = 0$ would have roots

$$\lambda_{\pm} = \frac{-v_1 \cdot v_2 \pm \sqrt{X}}{v_2^2} ,$$

where X was defined above.

The vectors v_1 and v_2 are either parallel (i.e., $v_1 = \alpha v_2$ for some real number α), in which case $X = 0$, or they are linearly independent. In this latter case they must span the two-dimensional vector space. It follows that $v^2(\lambda)$ takes on both positive and negative values, since the space contains both a spacelike and timelike vector. By continuity $v^2(\lambda)$ must vanish for one or more values of λ , so $v^2(\lambda)$ has real roots, and the above equation implies that $X \geq 0$. We conclude that $X \geq 0$. Moreover, $X = 0$ also implies the vectors are parallel. Indeed, for $X = 0$ the two roots are degenerate and $v^2(\lambda)$ has only one sign. Hence the vectors v_1 and v_2 are not linearly independent, so they must be parallel.

Solution by Alan Guth.

6.4 Stretched string and a nonrelativistic limit.

We are told to consider the string action in the form

$$S = -T_0 \int_{t_i}^{t_f} dt \int ds \sqrt{1 - \frac{v_\perp^2}{c^2}}. \quad (1)$$

The second integral here is simply an integral along the string; it is not appropriate to write limits of integration because the integration variable has not been chosen yet.

We describe the motion using the function $\vec{y}(t, x)$, where x is a (horizontal) coordinate along the equilibrium direction of the string and $\vec{y}$ represents transverse displacement. To perform the integral along the string in (1), we examine the contribution from the piece of string that stretches between x and $x + dx$. This piece is represented by the spatial vector $(dx, dX^2, \dots, dX^d) = (dx, d\vec{y})$. The length ds of this small piece of string is given by the Pythagorean theorem:

$$ds^2 = dx^2 + d\vec{y} \cdot d\vec{y}.$$

This implies that

$$ds = dx \sqrt{1 + \left(\frac{\partial \vec{y}}{\partial x}\right)^2} \simeq dx \left[1 + \frac{1}{2} \left(\frac{\partial \vec{y}}{\partial x}\right)^2\right], \quad (2)$$

where we used the small-oscillation approximation $|\frac{\partial \vec{y}}{\partial x}| \ll 1$. Additionally, since the string is almost horizontal everywhere and at all times ($|\frac{\partial \vec{y}}{\partial x}| \ll 1$, again) the perpendicular velocity is approximately equal to the vertical velocity $\frac{\partial \vec{y}}{\partial t}$, so we write

$$\vec{v}_\perp \simeq \frac{\partial \vec{y}}{\partial t}.$$

Since $|\vec{v}_\perp| \ll c$ (problem statement) we can approximate

$$\sqrt{1 - \frac{v_\perp^2}{c^2}} \simeq 1 - \frac{1}{2} \frac{v_\perp^2}{c^2} \simeq 1 - \frac{1}{2c^2} \left(\frac{\partial \vec{y}}{\partial t}\right)^2. \quad (3)$$

Using (2) and (3) in (1), we find

$$S \simeq -T_0 \int_{t_i}^{t_f} dt \int_0^a dx \left[1 + \frac{1}{2} \left(\frac{\partial \vec{y}}{\partial x}\right)^2\right] \left(1 - \frac{1}{2c^2} \left(\frac{\partial \vec{y}}{\partial t}\right)^2\right).$$

Since the terms in brackets and parentheses are of the form $1 + \epsilon$, with ϵ a small correction, multiplying out we only keep terms linear in the corrections

$$S \simeq -T_0 \int_{t_i}^{t_f} dt \int_0^a dx \left[1 + \frac{1}{2} \left(\frac{\partial \vec{y}}{\partial x}\right)^2 - \frac{1}{2c^2} \left(\frac{\partial \vec{y}}{\partial t}\right)^2\right].$$

The action thus becomes

$$S \simeq \int_{t_i}^{t_f} dt (-T_0 a) + \int_{t_i}^{t_f} dt \int_0^a dx \left[\frac{1}{2} \frac{T_0}{c^2} \left(\frac{\partial \vec{y}}{\partial t} \right)^2 - \frac{1}{2} T_0 \left(\frac{\partial \vec{y}}{\partial x} \right)^2 \right].$$

This can be compared with the action in equation (4.6.6). We have indeed obtained the action of a nonrelativistic string with tension T_0 and mass density $\mu_0 = T_0/c^2$. The additional constant corresponds to a potential energy $T_0 a$, namely tension times the length of the string.

6.5 Alternative derivation of the non-relativistic limit.

In this problem we work in the static gauge $X^0 \equiv ct = c\tau$ supplemented by the additional gauge condition $X^1 = x = a\sigma/\sigma_1$. The other embedding coordinates $X^i(\tau, \sigma)$, with $i = 2, 3, \dots, d$ can then be viewed as functions $X^i(t, x)$ of parameters t and x . They will be collectively denoted as $\vec{y}(t, x)$. The nonrelativistic approximation is based on

$$|\partial_x \vec{y}| \ll 1 \quad \text{and} \quad \left| \frac{1}{c} \partial_t \vec{y} \right| \ll 1. \quad (1)$$

We begin with the action

$$S = -\frac{T_0}{c} \int d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\partial_\sigma X \cdot \partial_\tau X)^2 - (\partial_\tau X)^2 (\partial_\sigma X)^2}.$$

Because of reparameterization invariance we can replace τ by t and σ by x to get:

$$S = -T_0 \int dt \int_0^a dx \sqrt{\left(\partial_x X \cdot \frac{1}{c} \partial_t X \right)^2 - \frac{1}{c^2} (\partial_t X)^2 (\partial_x X)^2}. \quad (2)$$

Noting that $\partial_x X^0 = 0$ and $\partial_x X^1 = 1$, we have

$$\begin{aligned} \frac{1}{c^2} (\partial_t X)^2 &= -1 + \frac{1}{c^2} (\partial_t \vec{y})^2, \\ (\partial_x X)^2 &= 1 + (\partial_x \vec{y})^2, \\ \left(\partial_x X \cdot \frac{1}{c} \partial_t X \right)^2 &= \left(\partial_x \vec{y} \cdot \frac{1}{c} \partial_t \vec{y} \right)^2. \end{aligned}$$

The term in the last line does not contribute since, on account of (1), it is a fourth power of a small number. The terms in the first two lines will contribute; each contains corrections quadratic in the small number. We can approximate S in (2) by action as

$$S \simeq -T_0 \int dt \int_0^a dx \sqrt{\left(1 - \frac{1}{c^2} (\partial_t \vec{y})^2 \right) \left(1 + (\partial_x \vec{y})^2 \right)}.$$

Multiplying out, keeping only quadratic terms, and using $\sqrt{1+\epsilon} \simeq 1 + \frac{1}{2}\epsilon$ we find the nonrelativistic action we derived previously in Problem 6.4:

$$S \simeq \int dt \int_0^a dx \left(-T_0 + \frac{1}{2} \frac{T_0}{c^2} (\partial_t \vec{y})^2 - \frac{1}{2} T_0 (\partial_x \vec{y})^2 \right).$$

6.6 Planar motion for open string with attached endpoints.

To solve this problem we must find an expression for $v_{\perp}^2$ in terms of y . Since the motion occurs in two spatial dimensions, the location of the string is described by

$$\vec{X} = (x, y(x, t)). \quad (1)$$

Associated with an infinitesimal dx we have the small string segment $d\vec{X}$ given by

$$d\vec{X} = (dx, y' dx) = (1, y') dx,$$

where prime denotes x derivative. The length ds of this piece of string is

$$ds = |d\vec{X}| = \sqrt{1 + y'^2} dx. \quad (2)$$

We now find

$$\frac{\partial \vec{X}}{\partial s} = \frac{dx}{ds} \frac{\partial \vec{X}}{\partial x} = \frac{1}{\sqrt{1 + y'^2}} (1, y'), \quad (3)$$

and we note that

$$\frac{\partial \vec{X}}{\partial t} = (0, \dot{y}). \quad (4)$$

Using the formula for $v_{\perp}^2$ from the notes

$$v_{\perp}^2 = \left(\frac{\partial \vec{X}}{\partial t} \right)^2 - \left(\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} \right)^2,$$

and evaluating the dot products using (3) and (4) we get

$$v_{\perp}^2 = \dot{y}^2 - \left(\frac{\dot{y}y'}{\sqrt{1 + y'^2}} \right)^2 = \dot{y}^2 - \frac{\dot{y}^2 y'^2}{1 + y'^2} = \frac{\dot{y}^2}{1 + y'^2}. \quad (5)$$

Finally, using (2) and (5), we find that the string Lagrangian becomes

$$\begin{aligned} L &= -T_0 \int ds \sqrt{1 - \frac{v_{\perp}^2}{c^2}} = -T_0 \int_0^a dx \sqrt{1 + y'^2} \sqrt{1 - \frac{1}{c^2} \frac{\dot{y}^2}{1 + y'^2}} \\ &= -T_0 \int_0^a dx \sqrt{1 + y'^2 - \frac{\dot{y}^2}{c^2}}. \end{aligned}$$

6.7 Time evolution of a closed circular string

Assume string is always circular, with radius $R(t)$.

Then $|\vec{V}_\perp| = \left| \frac{dR}{dt} \right|$ and $\int ds = 2\pi R$ so from (6.88)

$$S = -T_0 \int dt \, 2\pi R \sqrt{1 - \frac{1}{c^2} \left(\frac{dR}{dt} \right)^2} \quad \text{and we can treat}$$

this just like a dynamical system with one degree of freedom $R(t)$
and Lagrangian $L = -2\pi T_0 R \sqrt{1 - \frac{1}{c^2} \left(\frac{dR}{dt} \right)^2}$

The canonical momentum conjugate to R is

$$P = \frac{\partial L}{\partial \left(\frac{dR}{dt} \right)} = \frac{2\pi T_0}{c^2} \frac{R \frac{dR}{dt}}{\sqrt{1 - \frac{1}{c^2} \left(\frac{dR}{dt} \right)^2}}$$

Since $\frac{\partial L}{\partial R} = -2\pi T_0 \sqrt{1 - \frac{1}{c^2} \left(\frac{dR}{dt} \right)^2}$, the equation of motion is

$$\frac{dP}{dt} = \frac{\partial L}{\partial R}, \quad \frac{d}{dt} \left(\frac{R \frac{dR}{dt}}{\sqrt{1 - \frac{1}{c^2} \left(\frac{dR}{dt} \right)^2}} \right) + c^2 \sqrt{1 - \frac{1}{c^2} \left(\frac{dR}{dt} \right)^2} = 0$$

which eventually reduces to $R \frac{d^2 R}{dt^2} - \left(\frac{dR}{dt} \right)^2 + c^2 = 0 \quad (2.1)$

However, for a system with only one degree of freedom and with no explicit t dependence, we can solve by finding the energy integral,

$$H = P \frac{dR}{dt} - L = \text{constant}.$$

In this case,
$$H = \frac{2\pi T_0}{c^2} \frac{R \left(\frac{dR}{dt} \right)^2}{\sqrt{1 - \frac{1}{c^2} \left(\frac{dR}{dt} \right)^2}} + 2\pi T_0 R \sqrt{1 - \frac{1}{c^2} \left(\frac{dR}{dt} \right)^2}$$

$$= \frac{2\pi T_0 R}{\sqrt{1 - \frac{1}{c^2} \left(\frac{dR}{dt} \right)^2}} \quad \left(\text{which is certainly what we expect for the energy} \right)$$

At $t=0$, $R = R_0$ and $\frac{dR}{dt} = 0$ so $H = 2\pi T_0 R_0$.

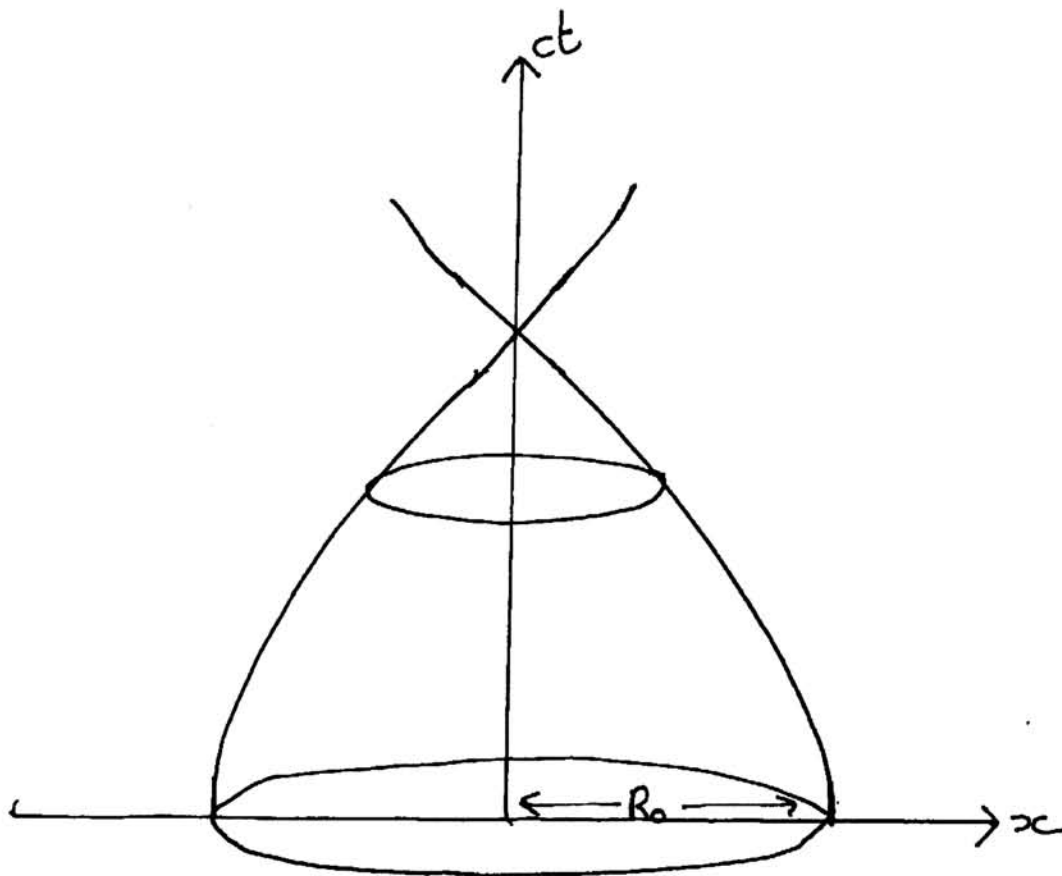
Hence
$$\frac{R}{\sqrt{1 - \frac{1}{c^2} \left(\frac{dR}{dt} \right)^2}} = R_0, \quad \left(\frac{dR}{dt} \right)^2 + c^2 \frac{R^2}{R_0^2} = c^2$$

which we recognize as the energy equation for simple harmonic motion, with the solution satisfying the initial conditions

$$R(t) = R_0 \cos \frac{ct}{R_0}, \quad \frac{dR}{dt} = -c \sin \frac{ct}{R_0}$$

(which you can check satisfies the equation of motion (2.1))

The radius goes to zero at time $\frac{\pi}{2} \frac{R_0}{c}$, at which time the string is imploding with the speed of light.



[If we had solved for x, y we would clearly get a solution which continued harmonic motion after $ct = \frac{\pi}{2} R_0$, returning to rest at $R = R_0$ when $ct = \pi R_0$]

6.8 Covariant analysis of open string endpoint motion

From (6.50) $P_\mu^\sigma = A \dot{X}'_\mu - B \dot{X}_\mu$

with $A = \frac{T_0 \dot{X}^2}{c \sqrt{(\dot{X} \cdot X')^2 - (\dot{X}^2)(X'^2)}}$, $B = \frac{T_0}{c} \frac{\dot{X} \cdot X'}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X}^2)(X'^2)}}$

$$P_\mu^\sigma P^{\sigma\mu} = A^2 (X')^2 + B^2 (\dot{X})^2 - 2AB (\dot{X} \cdot X')$$

$$= \left(\frac{T_0}{c}\right)^2 \frac{1}{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} \left\{ (\dot{X}^2)^2 (X'^2) + (\dot{X}^2)(\dot{X} \cdot X')^2 - 2(\dot{X}^2)(\dot{X} \cdot X')^2 \right\}$$

$$= -\left(\frac{T_0}{c}\right)^2 \dot{X}^2$$

At endpoints of open string, $P_\mu^\sigma = 0$ so $\dot{X}^2 = 0$
 which since endpoints are at fixed σ means they move at light-speed.

Added Note (BZ) $\dot{X}^2 = 0$ means that $\frac{\partial X^\mu}{\partial \tau}$ at

the endpoint is light-like. A light-like vector represents motion with speed of light.

The use of τ , as opposed to t , does not affect the discussion. If $\frac{\partial X^\mu}{\partial \tau}$ is light-like so is $\frac{\partial X^\mu}{\partial t}$ since $\frac{dt}{d\tau} \neq 0$

Since $\frac{\partial X^\mu}{\partial t} = (c, \vec{v})$, being light-like implies $|\vec{v}| = c$.

6.9 Hamiltonian density for relativistic strings

From (6.39) and (6.85), in the static gauge

$$S = \int \mathcal{L} d\sigma dt, \quad \mathcal{L} = -\frac{T_0}{c} \left[\left(\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma} \right)^2 + \left\{ c^2 - \left(\frac{\partial \vec{X}}{\partial t} \right)^2 \right\} \left(\frac{\partial \vec{X}}{\partial \sigma} \right)^2 \right]^{\frac{1}{2}}$$

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \vec{X}}{\partial t} \right)} = \frac{T_0}{c} \left\{ \frac{\partial \vec{X}}{\partial t} \left(\frac{\partial \vec{X}}{\partial \sigma} \right)^2 - \frac{\partial \vec{X}}{\partial \sigma} \left(\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma} \right) \right\} \left[\right]^{-\frac{1}{2}}$$

From (6.87) the quantity in the square bracket is $\left(\frac{ds}{d\sigma} \right)^2 (c^2 - v_{\perp}^2)$
and from (6.82) the quantity in the curly bracket is $\left(\frac{\partial \vec{X}}{\partial \sigma} \right)^2 \vec{v}_{\perp} = \left(\frac{ds}{d\sigma} \right)^2 \vec{v}_{\perp}$

Hence
$$\vec{p} = \frac{T_0}{c^2} \frac{\vec{v}_{\perp}}{\sqrt{1 - \frac{v_{\perp}^2}{c^2}}} \frac{ds}{d\sigma}$$

$$\mathcal{H} = \vec{p} \cdot \frac{\partial \vec{X}}{\partial t} - \mathcal{L} = \frac{T_0}{c^2} \frac{v_{\perp}^2}{\sqrt{1 - \frac{v_{\perp}^2}{c^2}}} \frac{ds}{d\sigma} + T_0 \sqrt{1 - \frac{v_{\perp}^2}{c^2}} \frac{ds}{d\sigma}$$

$$= \frac{T_0}{\sqrt{1 - \frac{v_{\perp}^2}{c^2}}} \frac{ds}{d\sigma}$$

$$\text{and } H = \int \mathcal{H} d\sigma = \int \frac{T_0 ds}{\sqrt{1 - \frac{v_{\perp}^2}{c^2}}}$$

Comparing to a point particle of rest mass m with $H = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$
we interpret as transverse motion with rest mass/unit length $= \frac{T_0}{c^2}$

6.10 Circular string in de Sitter space.

(a) The derivatives of the coordinates $X^\mu = \{ct, \vec{X}\}$ in static gauge $t = \tau$ are given by:

$$\begin{aligned}\dot{X}^0 &= \frac{\partial X^0}{\partial \tau} = c, & \dot{X}^i &= \frac{\partial X^i}{\partial \tau} = \frac{\partial X^i}{\partial t} . \\ X^{0'} &= \frac{\partial X^0}{\partial \sigma} = 0, & X^{i'} &= \frac{\partial X^i}{\partial \sigma} .\end{aligned}$$

With $g_{00} = -1$ and $g_{11} = g_{22} = g_{33} = e^{2Ht}$ we have

$$\begin{aligned}\dot{X} \cdot X' &= e^{2Ht} \frac{\partial X^i}{\partial t} \frac{\partial X^i}{\partial \sigma} = e^{2Ht} \dot{\vec{X}} \cdot \vec{X}' \\ \dot{X}^2 &= c^2 - e^{2Ht} \frac{\partial X^i}{\partial t} \frac{\partial X^i}{\partial t} = c^2 - e^{2Ht} \dot{\vec{X}}^2 \\ X'^2 &= e^{2Ht} \frac{\partial X^i}{\partial \sigma} \frac{\partial X^i}{\partial \sigma} = e^{2Ht} \vec{X}'^2 .\end{aligned}$$

In the above dots and squares indicate the ordinary vector dot product, $\vec{A} \cdot \vec{B} = \sum_i A^i B^i$.

Then the Nambu-Goto action

$$S = -\frac{T_0}{c} \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}$$

becomes

$$S = -\frac{T_0}{c} \int dt d\sigma e^{Ht} \sqrt{e^{2Ht} \left(\dot{\vec{X}} \cdot \vec{X}' \right)^2 + \left(c^2 - e^{2Ht} \dot{\vec{X}}^2 \right) \vec{X}'^2} .$$

(1)

(b) We now consider the circular string described in the statement of the problem. Note that even though σ has no units this is not a problem, as noted above (6.38).

$$\begin{aligned}X^{(0)} &= c\tau = ct, & X^{(1)} &= r \cos \sigma, & X^{(2)} &= r \sin \sigma \\ \dot{X}^{(0)} &= c, & \dot{X}^{(1)} &= -\dot{r} \sin \sigma, & \dot{X}^{(2)} &= \dot{r} \cos \sigma \\ X^{(0)'} &= 0, & X^{(1)'} &= -r \sin \sigma, & X^{(2)'} &= r \cos \sigma .\end{aligned}$$

Note that $\dot{\vec{X}} \cdot \vec{X}'$ actually vanishes:

$$\dot{\vec{X}} \cdot \vec{X}' = -\dot{r} \cos \sigma \sin \sigma + r \dot{r} \sin \sigma \cos \sigma = 0 .$$

Moreover, $\dot{\vec{X}}^2 = \dot{r}^2$ and $\vec{X}'^2 = r^2$. The action thus becomes

$$S = -\frac{T_0}{c} \int dt d\sigma e^{Ht} \sqrt{(c^2 - e^{2Ht} \dot{r}^2) r^2} = -2\pi T_0 \int dt e^{Ht} \sqrt{\left(1 - e^{2Ht} \frac{\dot{r}^2}{c^2}\right) r^2},$$

where we did the trivial integration over $\sigma \in [0, 2\pi]$. The Lagrangian is therefore given by

$$L(r, \dot{r}; t) = -2\pi T_0 r e^{Ht} \sqrt{1 - e^{2Ht} \frac{\dot{r}^2}{c^2}}.$$

As suggested in the problem, we now use the measured radius R of the string, given by $R = e^{Ht} r$. Then $\dot{R} = e^{Ht} H r + e^{Ht} \dot{r} = H R + e^{Ht} \dot{r}$, so $e^{Ht} \dot{r} = \dot{R} - H R$. Then

$$L(R, \dot{R}) = -2\pi T_0 R \sqrt{1 - \frac{1}{c^2} (\dot{R} - H R)^2}. \quad (2)$$

(c) In general $L = T - V$, and for $\dot{R} = 0$, $L = -V(R)$. So

$$V(R) = 2\pi T_0 R \sqrt{1 - \frac{1}{c^2} H^2 R^2}.$$

The units make sense because $[R] = L$ and therefore $T_0 R$ has units of energy. Note that inside the square root units also work. We have the Hubble length

$$R_H \equiv \frac{c}{H}.$$

and we can thus write the potential as

$$V(R) = (2\pi T_0 R_H) \frac{R}{R_H} \sqrt{1 - \frac{R^2}{R_H^2}}.$$

Note that for $R > R_H = c/H$ the potential becomes imaginary, which means that it is not a valid potential. This just means that for $R > R_H$, the velocity $\dot{R}$ cannot vanish. We introduce the constant ‘‘Hubble potential energy’’ V_H given by

$$V_H = 2\pi T_0 R_H.$$

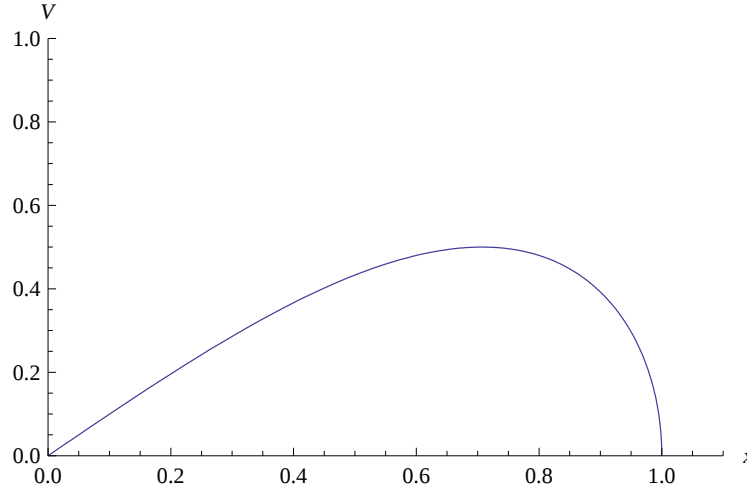
V_H is the flat-space potential energy of a circular string of Hubble radius. We then have that

$$\frac{V(R)}{V_H} = \frac{R}{R_H} \sqrt{1 - \frac{R^2}{R_H^2}}.$$

A plot of this potential

$$V(x) \equiv \frac{V(R)}{V_H} = x\sqrt{1-x^2}, \quad x \equiv \frac{R}{R_H},$$

is shown in the figure. The function $x\sqrt{1-x^2}$ has a maximum at $x = 1/\sqrt{2}$, therefore we



can get a static string at equilibrium for

$$R_c = \frac{1}{\sqrt{2}} R_H = \frac{1}{\sqrt{2}} \frac{c}{H}.$$

Since R_c corresponds to a maximum of the potential, the circular string is in unstable equilibrium.

(d) We begin by calculating the canonical momentum conjugate to R from the Lagrangian in Eq. (2). We get

$$\begin{aligned} P &= \frac{\partial L}{\partial \dot{R}} = -2\pi T_0 R \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1 - \frac{1}{c^2}(\dot{R} - HR)^2}} \left(-\frac{2}{c^2}\right) (\dot{R} - HR) \\ &= \frac{2\pi T_0}{c^2} \frac{R(\dot{R} - HR)}{\sqrt{1 - \frac{1}{c^2}(\dot{R} - HR)^2}}. \end{aligned}$$

$$L(R, \dot{R}) = -2\pi T_0 R \sqrt{1 - \frac{1}{c^2}(\dot{R} - HR)^2}. \quad (2)$$

Then

$$\begin{aligned}
H &= P\dot{R} - L \\
&= \frac{2\pi T_0}{c^2} \frac{R\dot{R}(\dot{R} - HR)}{\sqrt{1 - \frac{1}{c^2}(\dot{R} - HR)^2}} + 2\pi T_0 R \sqrt{1 - \frac{1}{c^2}(\dot{R} - HR)^2} \\
&= \frac{2\pi T_0}{c^2} \frac{1}{\sqrt{1 - \frac{1}{c^2}(\dot{R} - HR)^2}} \left\{ R\dot{R}(\dot{R} - HR) + R \left[c^2 - (\dot{R} - HR)^2 \right] \right\} \\
&= \frac{2\pi T_0}{c^2} \frac{R}{\sqrt{1 - \frac{1}{c^2}(\dot{R} - HR)^2}} \left\{ \dot{R}(\dot{R} - HR) + c^2 - \dot{R}(\dot{R} - HR) + HR(\dot{R} - HR) \right\} \\
&= \frac{2\pi T_0 R}{c^2} \frac{R}{\sqrt{1 - \frac{1}{c^2}(\dot{R} - HR)^2}} \left\{ c^2 + HR(\dot{R} - HR) \right\}
\end{aligned}$$

Finally,

$$H = 2\pi T_0 R \frac{1 + \frac{1}{c^2} HR(\dot{R} - HR)}{\sqrt{1 - \frac{1}{c^2}(\dot{R} - HR)^2}}.$$

Since $L(R, \dot{R})$ has no explicit time dependence, the energy function H is conserved.

Solution by Alan Guth and Barton Zwiebach

6.11 Open strings ending on D-branes of various dimensions.

(a) Recall that the position of the Dp -brane is specified by

$$x^a = 0, \text{ where } a = p+1, \dots, d. \quad (1)$$

The coordinates

$$x^i, \text{ where } i = 1, \dots, p \quad (2)$$

correspond to the directions on the Dp -brane. Since the strings are required to end on the brane,

$$X^a(t, \sigma=0) = 0, \quad (3)$$

where for simplicity we are focussing only on the end of the string at $\sigma = 0$. The boundary conditions require that for all allowed δX^μ we have

$$\delta X^\mu \mathcal{P}_\mu^\sigma = 0 \text{ at the endpoints.} \quad (4)$$

We now deduce that:

For $\mu = a$: Eq. (3) implies that X^a is fixed at the endpoints, so $\delta X^\mu = 0$, and therefore

There is no condition on $\mathcal{P}_a^\sigma$

For $\mu = i$: δX^i is free, so Eq. (4) implies that

$\mathcal{P}_i^\sigma = 0$ at the endpoint.

(5)

For $\mu = 0$: According to Eq. (6.57) of the textbook, we always have

$\mathcal{P}_0^\sigma = 0$ at the endpoint,

(6)

a relation which arises from varying X^0 at the endpoints.

Since in general we must have (6), we write it explicitly using (6.92), which is valid in static gauge:

$$\mathcal{P}_0^\sigma = -\frac{T_0}{c} \frac{\left(\frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial t} \right)}{\sqrt{1 - \frac{v_1^2}{c^2}}}. \quad (8)$$

The vanishing of this expression at the endpoints requires that

$$\frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial t} = 0, \text{ at the endpoints.} \quad (8')$$

As shown in (6.96), the expression for $\mathcal{P}^{\sigma\mu}$ in (6.91) simplifies by the use of (8') and becomes

$$\mathcal{P}^{\sigma\mu} = -T_0 \sqrt{1 - \frac{v^2}{c^2}} \frac{\partial X^\mu}{\partial s} \quad (8'')$$

Note also that (8') implies that $\vec{v} = \vec{v}_\perp$ at the endpoints.

(b) For a D0-brane, $p = 0$, so Eqs. (1) and (3) imply that $X^a(t, 0) = 0$ for $a = 1, \dots, d$. The case $\mu = i$ never occurs in this situation, so the only constraint on $\mathcal{P}_\mu^\sigma$ comes from Eq. (6). For a string ending on a D0-brane, all spatial coordinates at the endpoints are fixed, so

$$\frac{\partial \vec{X}}{\partial t} = 0, \quad (9)$$

implying in (8) that $\mathcal{P}_0^\sigma = 0$ at the endpoints, as claimed.

(c) For a D1-brane, Eqs. (1) and (3) imply that $X^a(t, 0) = 0$ for $a = 2, \dots, d$. The nontrivial constraints at the endpoints then take the form

$$\mathcal{P}_1^\sigma = 0 \quad (10a)$$

$$\mathcal{P}_0^\sigma = 0. \quad (10b)$$

Using (8'') one has

$$\mathcal{P}_1^\sigma = -T_0 \sqrt{1 - \frac{v_\perp^2}{c^2}} \frac{\partial X_1}{\partial s} = 0, \quad (11)$$

and $\mathcal{P}_0^\sigma$ is in Eq. (8).

Equation (11) can be satisfied either by requiring $v_\perp = c$, or $\partial X^1/\partial s = 0$. While either choice works for Eq. (11), the requirement that $\mathcal{P}_0^\sigma = 0$ restricts us to only the latter option. At the endpoints all the components of $\vec{X}$ except for X^1 are fixed, so Eqs. (8) and (10b) imply that

$$\mathcal{P}_0^\sigma = -\frac{T_0}{c} \frac{\left(\frac{\partial X^1}{\partial s} \frac{\partial X^1}{\partial t} \right)}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} = 0. \quad (12)$$

If $\partial X^1/\partial s \neq 0$, then the above equation implies that $\partial X^1/\partial t = 0$, since $\sqrt{1 - v_\perp^2/c^2}$ is never infinite. But the endpoint can only move along the brane, so $v_\perp = |\partial X^1/\partial t|$. Thus $\partial X^1/\partial s \neq 0$ excludes the possibility that $v_\perp = c$, so the only option consistent with both Eqs. (11) and (12) is

$\frac{\partial X^1}{\partial s} = 0 \text{ at the endpoints.}$

(13)

The constraints (11) and (12) are then satisfied for any value of the velocity. Since $\partial\vec{X}/\partial s$ is the tangent vector to the string, Eq. (13) implies that the tangent vector has no component in the x^1 direction, and hence it is orthogonal to the brane.

(d) For $p \geq 2$, Eqs. (11) and (12) are replaced by

$$\mathcal{P}_i^\sigma = -T_0 \sqrt{1 - \frac{v_\perp^2}{c^2}} \frac{\partial X_i}{\partial s} = 0 \quad (17)$$

and

$$\mathcal{P}_0^\sigma = -\frac{T_0}{c} \frac{\left(\frac{\partial X^i}{\partial s} \frac{\partial X^i}{\partial t}\right)}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} = 0. \quad (18)$$

where the index i takes on the values $1, \dots, p$, and the repeated index in Eq. (18) is summed over these values. There are then two ways to satisfy these equations:

(1) $v_\perp = c$: In this case Eq. (17) is automatically satisfied for all i , so the remaining constraint comes from Eq. (18). The resulting constraint is given by Eq. (8'). Although the sum in Eq. (8') includes all indices $1, \dots, d$, we know that the components of $\partial\vec{X}/\partial t$ have nonzero values only for the components along the brane, as required by Eq. (3). In words, Eq. (8') implies that the velocity of the string endpoint, $\partial\vec{X}/\partial t$, is orthogonal to the tangent vector along the string at its endpoint, $\partial\vec{X}/\partial s$.

(2) $v_\perp \neq c$: Then clearly Eq. (17) can be satisfied only if

$$\frac{\partial X^i}{\partial s} = 0 \text{ for all } i, \text{ at the endpoints.} \quad (19)$$

In words, Eq. (19) says that the tangent vector to the string at its endpoint, $\partial\vec{X}/\partial s$, has no components in the directions along the brane, $i = 1, \dots, p$, and therefore the string is orthogonal to the Dp -brane at its endpoint. This concludes the solution to the problem.

Comments: An interesting question arises if one specializes to static gauge, $X^0 = c\tau$, *before* one varies the action. In that case X^0 is no longer treated as a dynamical variable, so Eq. (6) is not obtained. As we now show, it can be derived from the other equations. To see this, note that $\mathcal{P}_\mu^\sigma$ is given explicitly by (Eq. (6.50)):

$$\mathcal{P}_\mu^\sigma = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')X'_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}. \quad (7)$$

From this expression one sees that

$$\mathcal{P}_\mu^\sigma \dot{X}^\mu = 0$$

identically all over the string. Expanding out the sum over the μ index

$$\mathcal{P}_0^\sigma \dot{X}^0 + \mathcal{P}_i^\sigma \dot{X}^i + \mathcal{P}_a^\sigma \dot{X}^a = 0.$$

At the endpoints we have $\mathcal{P}_i^\sigma = 0$ (from Eq. (5)), $\dot{X}^a = 0$ (from Eq. (3)), and $\dot{X}^0 \neq 0$. We thus learn that $\mathcal{P}_0^\sigma = 0$ at the endpoints. The boundary $\mathcal{P}_0^\sigma = 0$ will be **automatically** satisfied if the other boundary conditions are.

It is worth exploring a peculiar feature in the case of the D1-brane (part (c)). We argued that the equation $\mathcal{P}_0^\sigma = 0$ was redundant and could always be derived from the other endpoint constraint equations. In the above solution, however, Eq. (11) was not enough to give a unique answer, but instead Eq. (12) was necessary. Can we give a solution without using the vanishing of $\mathcal{P}_0^\sigma$?

Indeed, consider the general expression for $\mathcal{P}_\mu^\sigma$ in static gauge ((6.91)):

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2} \frac{\left(\frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial t}\right) \dot{X}^\mu + \left[c^2 - \left(\frac{\partial \vec{X}}{\partial t}\right)^2\right] \frac{\partial X^\mu}{\partial s}}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} . \quad (15)$$

If we now specialize to the D1-brane case, when $\partial \vec{X}/\partial t$ has only an X^1 component, we find

$$\mathcal{P}^{\sigma 1} = -\frac{T_0}{c^2} \frac{\left(\frac{\partial X^1}{\partial s} \frac{\partial X^1}{\partial t}\right) \frac{\partial X^1}{\partial t} + \left[c^2 - \left(\frac{\partial X^1}{\partial t}\right)^2\right] \frac{\partial X^1}{\partial s}}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} .$$

Cancelling out terms,

$$\mathcal{P}^{\sigma 1} = -T_0 \frac{1}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \frac{\partial X^1}{\partial s} . \quad (16)$$

Requiring this quantity to vanish gives us immediately the constraint condition, Eq. (13), with no constraint on v , as we expected.

Solution by Jeffrey Goldstone, Alan Guth, and Barton Zwiebach!

7.1 Open strings with free endpoints

From (7.44) (in which the only boundary condition imposed is $\frac{\partial \vec{X}}{\partial \sigma} = 0$ at $\sigma = 0$)

$$\vec{X}(t, \sigma) = \frac{1}{2} \{ \vec{F}(ct + \sigma) + \vec{F}(ct - \sigma) \}$$

Since this gives $\vec{X}(t, 0) = \vec{F}(ct)$ we can write it as

$$\vec{X}(t, \sigma) = \frac{1}{2} \{ \vec{X}(t + \frac{\sigma}{c}, 0) + \vec{X}(t - \frac{\sigma}{c}, 0) \} \quad (1.1)$$

a) A hyperplane is defined by a set of equations

$$\vec{a}_r \cdot \vec{x} = b_r$$

If $\sigma = 0$ always lies in this hyperplane, $\vec{a}_r \cdot \vec{X}(t, 0) = b_r$ for all t and then from (1.1), $\vec{a}_r \cdot \vec{X}(t, \sigma) = b_r$ for all t and σ i.e. the whole string lies in the hyperplane.

b) From (1.1),

$$|\vec{X}(t, \sigma) - \vec{X}_0| = \frac{1}{2} | \{ \vec{X}(t + \frac{\sigma}{c}, 0) - \vec{X}_0 \} + \{ \vec{X}(t - \frac{\sigma}{c}, 0) - \vec{X}_0 \} |$$

$$\leq \frac{1}{2} \{ |\vec{X}(t + \frac{\sigma}{c}, 0) - \vec{X}_0| + |\vec{X}(t - \frac{\sigma}{c}, 0) - \vec{X}_0| \}$$

Hence $|\vec{X}(t, 0) - \vec{X}_0| \leq R$ for all t ,

$$|\vec{X}(t, \sigma) - \vec{X}_0| \leq R \text{ for all } t, \sigma$$

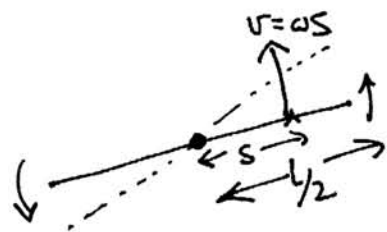
c) A convex subspace is a region S with the property that if $\vec{x}$ and $\vec{y}$ lie in S , so does $(1-\lambda)\vec{x} + \lambda\vec{y}$ for $0 \leq \lambda \leq 1$ (i.e. all the points between $\vec{x}$ and $\vec{y}$ on the line joining them). (1.1) immediately tells us that if $\vec{X}(t + \frac{\sigma}{c}, 0)$ and $\vec{X}(t - \frac{\sigma}{c}, 0)$ lie in S , so does $\vec{X}(t, \sigma)$.

d) The condition that defines the parameter σ is (7.37)

$$\frac{ds}{d\sigma} = \sqrt{1 - \frac{v_{\perp}^2}{c^2}}$$

$$\text{Hence } l = \int_0^{\sigma_1} ds = \int_0^{\sigma_1} \sqrt{1 - \frac{v_{\perp}^2}{c^2}} d\sigma$$

7.2 Rigidly rotating string



a) $\mathcal{E} = \frac{T_0}{\sqrt{1 - \frac{v_\perp^2}{c^2}}}$

For rigid rotation, $v_\perp = \omega s$

At the ends, $s = \frac{L}{2}$, $v_\perp = c$ so $\omega = \frac{2c}{L}$,

$v_\perp = \frac{2cs}{L}$ and $\mathcal{E}(s) = \frac{T_0}{\sqrt{1 - \frac{4s^2}{L^2}}}$

Total energy is $\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{T_0 ds}{\sqrt{1 - \frac{4s^2}{L^2}}}$

With $s = \frac{L}{2} \sin \theta$, energy = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{T_0 L}{2} \frac{\cos \theta d\theta}{\cos \theta} = \frac{\pi}{2} T_0 L$

b) Average energy density is $\frac{\pi}{2} T_0$, equals local energy density where $\sqrt{1 - \frac{4s^2}{L^2}} = \frac{2}{\pi}$, $s/L = \pm \frac{1}{2} \sqrt{1 - \frac{4}{\pi^2}} = \pm 0.3856$

c) $\hat{E}(s) = \int_{-s}^s \frac{T_0 ds'}{\sqrt{1 - \frac{4s'^2}{L^2}}} = T_0 L \theta$ where $s = \frac{L}{2} \sin \theta$

and $\hat{E}(s)/\hat{E}(\frac{L}{2}) = \frac{2\theta}{\pi}$

This is $1/2$ for $\theta = \frac{\pi}{4}$, $\sin \theta = \frac{1}{\sqrt{2}} = 0.7071$, $s = 0.7071 (L/2)$

It is 0.9 for $\theta = 0.45\pi$, $\frac{s}{(L/2)} = \sin 0.45\pi = 0.988$

"the $\approx 1\%$ of string near the endpoints carries about 10% of the energy"
(B.2)

7.3 Time evolution of an initially static closed string

a) From (7.33) we get

$$\vec{X}(t, \sigma) = \frac{1}{2} \left\{ \vec{F}(ct + \sigma) + \vec{G}(ct - \sigma) \right\}$$

$$\begin{aligned} \frac{\partial \vec{X}}{\partial t}(t=0, \sigma) &= \frac{c}{2} \left\{ \vec{F}'(\sigma) + \vec{G}'(-\sigma) \right\} \\ &= \frac{c}{2} \frac{d}{d\sigma} \left\{ \vec{F}(\sigma) - \vec{G}(-\sigma) \right\} \end{aligned}$$

so if $\frac{\partial \vec{X}}{\partial t} = 0$ at $t=0$, $\vec{G}(\sigma) = \vec{F}(-\sigma) + \vec{a}$,

$$\vec{X}(t, \sigma) = \frac{1}{2} \left\{ \vec{F}(ct + \sigma) + \vec{F}(\sigma - ct) + \vec{a} \right\}$$

and we can absorb $\frac{\vec{a}}{2}$ into $\vec{F}$ to get

$$\vec{X}(t, \sigma) = \frac{1}{2} \left\{ \vec{F}(\sigma + ct) + \vec{F}(\sigma - ct) \right\}$$

$$[\vec{X}'(0, \sigma) = \vec{F}'(\sigma)]$$

Parametrization conditions are

$$\left(\frac{\partial \vec{X}}{\partial \sigma} \pm \frac{1}{c} \frac{\partial \vec{X}}{\partial t} \right)^2 = 1$$

$$\frac{\partial \vec{X}}{\partial \sigma} = \frac{1}{2} \left\{ \vec{F}'(\sigma + ct) + \vec{F}'(\sigma - ct) \right\}; \quad \frac{\partial \vec{X}}{\partial t} = \frac{1}{2} \left\{ \vec{F}'(\sigma + ct) - \vec{F}'(\sigma - ct) \right\}$$

so $\frac{\partial \vec{X}}{\partial \sigma} \pm \frac{1}{c} \frac{\partial \vec{X}}{\partial t} = \vec{F}'(\sigma \pm ct)$ and to satisfy

conditions we must have $|\vec{F}'(u)| = 1$, all u .

b) We need $\vec{X}(t, \sigma + \sigma_1) = \vec{X}(t, \sigma)$

which requires $\vec{F}(u + \sigma_1) = \vec{F}(u)$

i.e. $\vec{F}$ must be periodic with period σ_1 .

c) We can set $t_0 = 0$ (to use previous formulae).

At $t = 0$, γ is given by $\vec{X} = \vec{F}(\sigma)$,
 $0 \leq \sigma < \sigma_1$ and since $|\vec{F}'| = 1$, σ is in fact the length
 along γ , so $\sigma_1 = L$.

$$\text{At } t_p, \quad \vec{X}(\sigma) = \frac{1}{2} \{ \vec{F}(\sigma + ct_p) + \vec{F}(\sigma - ct_p) \}$$

The only things we know about $\vec{F}$ are $|\vec{F}'| = 1$ and $\vec{F}(\sigma + \sigma_1) = \vec{F}(\sigma)$

We can use the periodicity when $ct_p = \frac{\sigma_1}{2}$, since

$$\text{then } \vec{X}(\sigma) = \frac{1}{2} \{ \vec{F}(\sigma + \frac{\sigma_1}{2}) + \vec{F}(\sigma - \frac{\sigma_1}{2}) \} = \vec{F}(\sigma + \frac{\sigma_1}{2})$$

which means that as σ goes from 0 to σ_1 , $\vec{X}(\sigma)$ runs
 along γ but from $\sigma = \frac{\sigma_1}{2}$ to $\sigma = \frac{3\sigma_1}{2}$.

$$\text{In fact we have } \vec{X}(\frac{\sigma_1}{2c}, \sigma) = \vec{X}(0, \sigma + \frac{\sigma_1}{2})$$

(and at $t = \frac{\sigma_1}{2c}$, $\frac{\partial \vec{X}}{\partial t} = 0$ again). Also $t_p = \frac{L}{2c}$

d) We first need to express the initial curve as $\vec{X} = \vec{F}(\sigma)$
 with $\sigma = S$. Step 1 is to calculate $S(\lambda)$ by

$$S(\lambda) = \int_0^\lambda d\lambda' \left\{ \left(\frac{dx}{d\lambda'} \right)^2 + \left(\frac{dy}{d\lambda'} \right)^2 \right\}^{\frac{1}{2}}.$$

Step 2 is to invert to express $\lambda = \lambda(s)$ and then to
 express $[x(\lambda), y(\lambda)]$ as $[X(s), Y(s)] = \vec{F}(s)$

$$\text{Now we can compute } \vec{X}(t, \sigma) \approx \frac{1}{2} \{ \vec{F}(\sigma + ct) + \vec{F}(\sigma - ct) \}$$

7.4 Relativistic jumping rope

$$\vec{X}(t, \sigma) = \vec{x}_1 + \frac{1}{2} \{ \vec{F}(ct + \sigma) - \vec{F}(ct - \sigma) \}$$

a) $\vec{X}(t, \sigma_1) = \vec{x}_2$, all t , gives

$$\vec{F}(ct + \sigma_1) - \vec{F}(ct - \sigma_1) = 2(\vec{x}_2 - \vec{x}_1)$$

i.e. $\vec{F}(u + 2\sigma_1) = \vec{F}(u) + 2(\vec{x}_2 - \vec{x}_1)$

This means that $\vec{F}(u) = \vec{G}(u) + (\vec{x}_2 - \vec{x}_1) \frac{u}{\sigma_1}$

with $\vec{G}(u + 2\sigma_1) = \vec{G}(u)$

b) $\frac{\partial \vec{X}}{\partial \sigma} \pm \frac{1}{c} \frac{\partial \vec{X}}{\partial t} = \vec{F}'(ct \pm \sigma)$

so we must have $|\vec{F}'(u)| = 1$

c) $\left. \frac{\partial \vec{X}}{\partial \sigma} \right|_{\sigma=0} = \vec{F}'(ct)$ and we are told this

is the unit vector $(\sin \gamma \cos \omega t, \sin \gamma \sin \omega t, \cos \gamma)$

so that $\vec{F}'(u) = (\sin \gamma \cos \frac{\omega}{c} u, \sin \gamma \sin \frac{\omega}{c} u, \cos \gamma)$

(assuming a steady rotation about the z axis, angular velocity ω)

d) $\vec{F}'(u)$ must have period $2\sigma_1$ so we take $\sigma_1 = \frac{\pi c}{\omega}$

$$\vec{F}(2\sigma_1) - \vec{F}(0) = \int_0^{2\sigma_1} \vec{F}'(u) du = 2(\vec{x}_2 - \vec{x}_1)$$

$$= (0, 0, 2L_0) \text{ gives } \frac{2\pi c}{\omega} \cos \gamma = 2L_0$$

so $\sigma_1 = L_0 / \cos \gamma$ and $\omega = \frac{\pi c}{L_0} \cos \gamma$

$$e) \quad \vec{F}(u) = \left[\frac{c}{\omega} \sin \gamma \sin \frac{\omega}{c} u, -\frac{c}{\omega} \sin \gamma \cos \frac{\omega}{c} u, u \cos \gamma \right]$$

$$\text{so } \vec{X}(t, \sigma) = [X, Y, Z],$$

$$X = \frac{c}{\omega} \sin \gamma \left\{ \frac{1}{2} \sin \omega \left(t + \frac{\sigma}{c} \right) - \frac{1}{2} \sin \omega \left(t - \frac{\sigma}{c} \right) \right\}$$

$$Y = -\frac{c}{\omega} \sin \gamma \left\{ \frac{1}{2} \cos \omega \left(t + \frac{\sigma}{c} \right) - \frac{1}{2} \cos \omega \left(t - \frac{\sigma}{c} \right) \right\}$$

$$Z = \sigma \cos \gamma$$

$$\text{so } X = \frac{L_0}{\pi} \tan \gamma \sin \frac{\pi \sigma}{\sigma_1} \cos \omega t$$

$$Y = \frac{L_0}{\pi} \tan \gamma \sin \frac{\pi \sigma}{\sigma_1} \sin \omega t$$

$$Z = L_0 \frac{\sigma}{\sigma_1}$$

$$\text{and at } t=0, \quad Y=0, \quad X = \frac{L_0}{\pi} \tan \gamma \sin \frac{\pi Z}{L_0}$$

$$\text{and rope rotates with period } \frac{2L_0}{c \cos \gamma}$$

f) Since $Z = L_0 \frac{\sigma}{\sigma_1}$ and energy is distributed uniformly in σ , the energy is also distributed uniformly along the z -direction.

$$\text{In fact } dz = \frac{L_0}{\sigma_1} d\sigma = \frac{L_0}{(E/T_0)} \frac{dE}{T_0} \quad (7.24)$$

$$\rightarrow \frac{dE}{dz} = \frac{E}{L_0} \quad (\text{B.Z.})$$

7.5 Planar motion for an open string with fixed endpoints.

(a) We check first that $|\vec{F}'(u)|^2 = 1$. We readily confirm that it holds:

$$|\vec{F}'(u)|^2 = \cos^2\left(\gamma \cos \frac{\pi u}{\sigma_1}\right) + \sin^2\left(\gamma \cos \frac{\pi u}{\sigma_1}\right) = 1. \quad (1)$$

The other condition is

$$\vec{F}(u + 2\sigma_1) - \vec{F}(u) = (2a, 0). \quad (2)$$

Differentiating both sides with respect to u gives the periodicity condition

$$\vec{F}'(u + 2\sigma) = \vec{F}'(u),$$

which holds since $\vec{F}'(u)$ is a function of $\cos \frac{\pi u}{\sigma_1}$ only. In full detail, condition (2) requires

$$\int_0^{2\sigma_1} du \vec{F}'(u) = (2a, 0), \quad (3)$$

which implies a relation among the constants a , γ , and σ_1 , to be discussed in part (d).

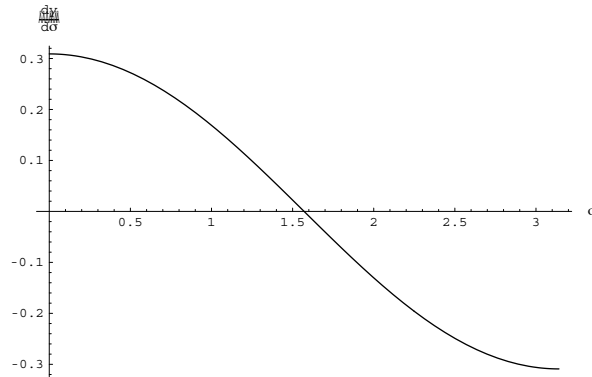
(b) We have

$$\vec{X}'(0, \sigma) = \left(\frac{dx}{d\sigma}, \frac{dy}{d\sigma} \right) = \frac{1}{2} \left(\vec{F}'(\sigma) + \vec{F}'(-\sigma) \right) = \vec{F}'(\sigma),$$

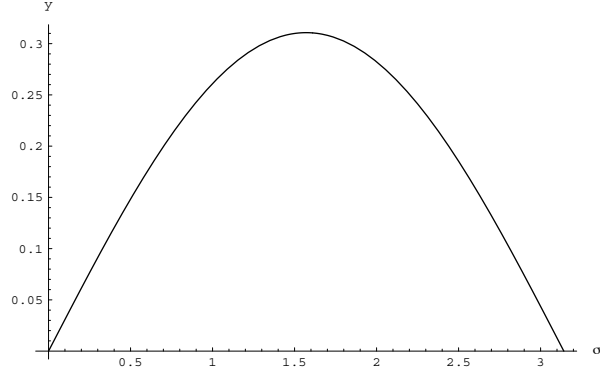
where we used $\vec{F}'(u) = \vec{F}'(-u)$. This gives

$$\frac{dy}{d\sigma} = \sin\left(\gamma \cos \frac{\pi \sigma}{\sigma_1}\right).$$

Below we plot $dy/d\sigma$ for $\gamma = \pi/10$ and $\sigma_1 = \pi$.



Integrating $dy/d\sigma$ gives $y(\sigma)$, as shown in the following figure:



(c) We have

$$\vec{X}'(t, 0) = \frac{1}{2} \left(\vec{F}'(ct + 0) + \vec{F}'(ct - 0) \right) = \vec{F}'(ct) = \left(\cos \left[\gamma \cos \frac{\pi ct}{\sigma_1} \right], \sin \left[\gamma \cos \frac{\pi ct}{\sigma_1} \right] \right).$$

Note that $\vec{X}'(t, 0)$ is just the unit tangent vector along the path of the string at its left endpoint. The angle θ between the tangent vector and horizontal is given by

$$\theta(t) = \gamma \cos \frac{\pi ct}{\sigma_1},$$

so that $\theta(t = 0) = \gamma$. Thus γ is the initial and maximum angle of the end of the string.

(d) The first component of the condition in (3) gives

$$\int_0^{2\sigma_1} du \vec{F}'_x(u) = 2a.$$

where $\vec{F}' = (\vec{F}'_x, \vec{F}'_y)$. The second component of equation (3) is automatically satisfied (check this!). More explicitly, the above equation gives

$$\int_0^{2\sigma_1} du \cos \left[\gamma \cos \frac{\pi u}{\sigma_1} \right] = 2a.$$

Letting $\pi u / \sigma_1 = x$, we rewrite the above equation as

$$\frac{a}{\sigma_1} = \frac{1}{2\pi} \int_0^{2\pi} dx \cos(\gamma \cos x). \quad (4)$$

For $\gamma \ll 1$, we expand the integrand in powers of γ :

$$\frac{a}{\sigma_1} = \frac{1}{2\pi} \int_0^{2\pi} dx \left(1 - \frac{\gamma^2}{2} \cos^2 x + \mathcal{O}(\gamma^4) \right) = 1 - \frac{\gamma^2}{4} + \mathcal{O}(\gamma^4).$$

This is the approximate relation between a, σ_1 and γ , for small γ .

(e) With the integral representation $J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} dx \cos(x \cos \theta)$ for the Bessel function, we immediately recognize that (4) is just equivalent to $a/\sigma_1 = J_0(\gamma)$.

7.6 Planar motion (continued) and the formation of cusps.

(a) We have that

$$\vec{X}'(t, \sigma) = \frac{\partial \vec{X}}{\partial \sigma} = \frac{1}{2} \left(\vec{F}'(ct + \sigma) + \vec{F}'(ct - \sigma) \right), \quad (1)$$

and

$$\vec{F}'(u) = \left(\cos \left[\gamma \cos \frac{\pi u}{\sigma_1} \right], \sin \left[\gamma \cos \frac{\pi u}{\sigma_1} \right] \right).$$

Consider the x component of equation (1):

$$\vec{X}'_x = \frac{1}{2} \cos \left[\gamma \cos \frac{\pi(ct + \sigma)}{\sigma_1} \right] + \frac{1}{2} \cos \left[\gamma \cos \frac{\pi(ct - \sigma)}{\sigma_1} \right]$$

It is convenient to define

$$\gamma \cos \frac{\pi(ct \pm \sigma)}{\sigma_1} \equiv \beta \mp \alpha, \quad \text{so} \quad \beta = \gamma \cos \frac{\pi ct}{\sigma_1} \cos \frac{\pi \sigma}{\sigma_1}, \quad \alpha = \gamma \sin \frac{\pi ct}{\sigma_1} \sin \frac{\pi \sigma}{\sigma_1}. \quad (2)$$

With this notation,

$$\vec{X}'_x = \frac{1}{2} (\cos(\beta - \alpha) + \cos(\beta + \alpha)) = \cos \alpha \cos \beta.$$

Similarly,

$$\vec{X}'_y = \frac{1}{2} (\sin(\beta - \alpha) + \sin(\beta + \alpha)) = \cos \alpha \sin \beta.$$

Assembling together these results,

$$\vec{X}'(t, \sigma) = \cos \alpha \left(\cos \beta, \sin \beta \right) = \cos \left(\gamma \sin \frac{\pi ct}{\sigma_1} \sin \frac{\pi \sigma}{\sigma_1} \right) \left(\cos \beta, \sin \beta \right). \quad (3)$$

Note that when $ct = \sigma_1/2$, we obtain $\beta = 0$ so that

$$\vec{X}' = \cos \left(\gamma \sin \frac{\pi}{2} \sin \frac{\pi \sigma}{\sigma_1} \right) (1, 0) = \cos \left(\gamma \sin \frac{\pi \sigma}{\sigma_1} \right) (1, 0).$$

Since $\vec{X}'$ points along the x axis for all σ , the string is horizontal.

(b) We have

$$\frac{1}{c} \frac{\partial \vec{X}}{\partial t} = \frac{1}{2} \left(\vec{F}'(ct + \sigma) - \vec{F}'(ct - \sigma) \right).$$

There is only one sign difference between this expression and that for $\vec{X}'$. Thus we get immediately that

$$\frac{1}{c} \frac{\partial \vec{X}_x}{\partial t} = \frac{1}{2} (\cos(\beta - \alpha) - \cos(\beta + \alpha)) = \sin \alpha \sin \beta,$$

and similarly,

$$\frac{1}{c} \frac{\partial \vec{X}_y}{\partial t} = \frac{1}{2} \left(\sin(\beta - \alpha) - \sin(\beta + \alpha) \right) = -\cos \beta \sin \alpha.$$

Together, the above equations give

$$\frac{1}{c} \frac{\partial \vec{X}}{\partial t} = \sin \alpha \left(\sin \beta, -\cos \beta \right).$$

It follows that

$$\left| \frac{1}{c} \frac{\partial \vec{X}}{\partial t} \right| = |\sin \alpha| = \left| \sin \left(\gamma \sin \frac{\pi ct}{\sigma_1} \sin \frac{\pi \sigma}{\sigma_1} \right) \right|,$$

which is what we wanted to show.

At $t = 0$ the velocity is clearly zero. When $\gamma < \pi/2$, we have

$$-\frac{\pi}{2} < \gamma \sin \frac{\pi ct}{\sigma_1} \sin \frac{\pi \sigma}{\sigma_1} = \alpha < \frac{\pi}{2},$$

so $|\sin \alpha| < 1$ and the velocity of any point on the string at any time is strictly smaller than c . For $\gamma = \pi/2$ and $ct = \sigma_1/2$ we have

$$\left| \frac{1}{c} \frac{\partial \vec{X}}{\partial t} \right| = \left| \sin \left(\frac{\pi}{2} \sin \frac{\pi \sigma}{\sigma_1} \right) \right|.$$

At the midpoint of the string, $\sigma = \sigma_1/2$ this gives $|\frac{1}{c} \frac{\partial \vec{X}}{\partial t}| = |\sin \frac{\pi}{2}| = 1$.

(c) Setting $\gamma = \sqrt{2}(\pi/2)$ and $ct = \sigma_1/4$ we have

$$\left| \frac{1}{c} \frac{\partial \vec{X}}{\partial t} \right| = \left| \sin \left(\sqrt{2} \frac{\pi}{2} \sin \frac{\pi}{4} \sin \frac{\pi \sigma}{\sigma_1} \right) \right| = \left| \sin \left(\frac{\pi}{2} \sin \frac{\pi \sigma}{\sigma_1} \right) \right|.$$

This is the same formula that we found for the case $\gamma = \pi/2$ and $ct = \sigma_1/2$, so we again find that only one point, $\sigma = \sigma_1/2$ has reached the speed of light.

Setting $\gamma = \sqrt{2}(\pi/2)$ and $ct = \sigma_1/3$ we have

$$\left| \frac{1}{c} \frac{\partial \vec{X}}{\partial t} \right| = \left| \sin \left(\sqrt{2} \frac{\pi}{2} \sin \frac{\pi}{3} \sin \frac{\pi \sigma}{\sigma_1} \right) \right| = \left| \sin \left(\sqrt{\frac{2}{3}} \frac{\pi}{2} \sin \frac{\pi \sigma}{\sigma_1} \right) \right|.$$

This equals one when

$$\sin \frac{\pi \sigma}{\sigma_1} = \sqrt{\frac{2}{3}},$$

which occurs at two values σ_l, σ_r of σ :

$$\frac{\pi \sigma_l}{\sigma_1} = \sin^{-1} \sqrt{\frac{2}{3}} \quad \text{and} \quad \frac{\pi \sigma_r}{\sigma_1} = \pi - \sin^{-1} \sqrt{\frac{2}{3}},$$

To see that these points are cusps, we examine $\vec{X}'$ near $\sigma = \sigma_l$ using equation (3):

$$\vec{X}'\left(t = \frac{\sigma_1}{3c}, \sigma\right) = \cos\left(\frac{\pi}{2}\sqrt{\frac{3}{2}}\sin\frac{\pi\sigma}{\sigma_1}\right)\left(\cos\beta, \sin\beta\right). \quad (4)$$

Here

$$\beta = \gamma \cos\frac{\pi ct}{\sigma_1} \cos\frac{\pi\sigma}{\sigma_1} = \frac{\pi}{2}\frac{1}{\sqrt{2}}\cos\frac{\pi\sigma}{\sigma_1}.$$

We note that at $\sigma = \sigma_l$ the value β_l of β is completely regular:

$$\beta_l = \frac{\pi}{2}\frac{1}{\sqrt{6}}.$$

We can therefore set $\beta = \beta_l$ in equation (4). The prefactor on the right-hand side of (4) is more delicate. Putting $\sigma = \sigma_l + \delta\sigma$ we have to lowest order in $\delta\sigma$ that

$$\sin\frac{\pi(\sigma_l + \delta\sigma)}{\sigma_1} = \sqrt{\frac{2}{3}} + \frac{1}{\sqrt{3}}\frac{\pi}{\sigma_1}\delta\sigma + \mathcal{O}(\delta\sigma^2).$$

Therefore,

$$\frac{\pi}{2}\sqrt{\frac{3}{2}}\sin\frac{\pi(\sigma_l + \delta\sigma)}{\sigma_1} = \frac{\pi}{2} + \frac{1}{\sqrt{2}}\frac{\pi}{\sigma_1}\delta\sigma + \mathcal{O}(\delta\sigma^2),$$

which implies that

$$\cos\left(\frac{\pi}{2}\sqrt{\frac{3}{2}}\sin\frac{\pi(\sigma_l + \delta\sigma)}{\sigma_1}\right) = -\frac{1}{\sqrt{2}}\frac{\pi}{\sigma_1}\delta\sigma + \mathcal{O}(\delta\sigma^2).$$

Finally, back in (4), we have

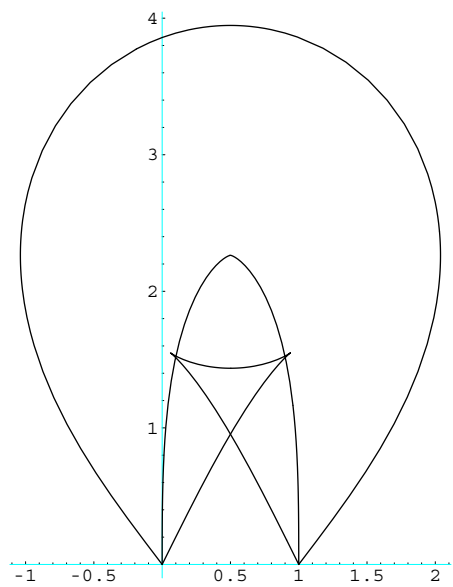
$$\vec{X}'\left(t = \frac{\sigma_1}{3c}, \sigma_l + \delta\sigma\right) = \delta\sigma\left(-\frac{1}{\sqrt{2}}\frac{\pi}{\sigma_1}\right)\left(\cos\beta_l, \sin\beta_l\right) = \delta\sigma(-\vec{V}_l).$$

Near σ_l the tangent vector $\vec{X}'$ to the string is equal to a constant vector $\vec{V}_l$ times $\delta\sigma$. For σ slightly smaller than σ_1 and growing we are approaching the cusp, $\delta\sigma < 0$, and the tangent points along $+\vec{V}_l$. For σ slightly larger than σ_1 we are moving away from the cusp, $\delta\sigma > 0$ and the tangent points along $(-\vec{V}_l)$. The tangent to the string reverses direction at σ_l , and it does so discontinuously. This is a cusp. Note that β_l is simply the angle that the cusp makes with respect to the horizontal.

(d) Setting $a = 1$, requires

$$\sigma_1 = \frac{1}{J_0\left(\frac{\pi\sqrt{2}}{2}\right)} \simeq 10.154868.$$

For these values of σ_1 and γ we plot the strings that occur for $ct = 0, \sigma_1/4$, and $\sigma_1/3$,



7.7 Cusps in the evolution of closed strings.

The closed string motion is described by

$$\vec{X}(t, \sigma) = \frac{1}{2} \left(\vec{F}(u) + \vec{G}(v) \right), \quad u = ct + \sigma, \quad v = ct - \sigma. \quad (1)$$

where $\vec{F}'(u)$ and $\vec{G}'(v)$ are unit vectors and are periodic functions of their arguments with period σ_1 . At the point of the cusp, $u = u_0$ and $v = v_0$, we have

$$\vec{F}'(u_0) = \vec{G}'(v_0). \quad (2)$$

The quantities u_0 and v_0 define a time t_0 and position σ_0 .

(a) Using $\frac{\partial u}{\partial \sigma} = 1$ and $\frac{\partial v}{\partial \sigma} = -1$ it is easy to see that

$$\frac{\partial^n \vec{F}(u)}{\partial \sigma^n} = \vec{F}^{(n)}(u), \quad \frac{\partial^n \vec{G}(v)}{\partial \sigma^n} = (-1)^n \vec{G}^{(n)}(v).$$

We can approximate $\vec{X}(t_0, \sigma)$ near the cusp by a Taylor expansion (t_0 is fixed, so only derivatives with respect to σ enter in the expansion):

$$\vec{X}(t_0, \sigma) = \vec{X}(t_0, \sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n \vec{X}}{\partial \sigma^n} \bigg|_{t_0, \sigma_0} (\sigma - \sigma_0)^n.$$

We then note that

$$\frac{\partial^n \vec{X}}{\partial \sigma^n} \bigg|_{t_0, \sigma_0} = \frac{1}{2} \left(\frac{\partial^n \vec{F}(u)}{\partial \sigma^n} + \frac{\partial^n \vec{G}(v)}{\partial \sigma^n} \right) \bigg|_{u_0, v_0} = \frac{1}{2} \left(\vec{F}^{(n)}(u_0) + (-1)^n \vec{G}^{(n)}(v_0) \right).$$

Using the last two equations we find

$$\begin{aligned} \vec{X}(t_0, \sigma) &= \frac{1}{2} \left(\vec{F}(u_0) + \vec{G}(v_0) \right) + \frac{1}{2} \left(\vec{F}'(u_0) - \vec{G}'(v_0) \right) (\sigma - \sigma_0) \\ &\quad + \frac{1}{4} \left(\vec{F}''(u_0) + \vec{G}''(v_0) \right) (\sigma - \sigma_0)^2 + \frac{1}{12} \left(\vec{F}'''(u_0) - \vec{G}'''(v_0) \right) (\sigma - \sigma_0)^3 + \dots \end{aligned}$$

The first term in the expansion vanishes because we have placed the cusp at the origin and the second because of the intersection of the paths. Therefore,

$$\vec{X}(t_0, \sigma) = \frac{1}{2} \vec{T}(\sigma - \sigma_0)^2 + \frac{1}{3!} \vec{R}(\sigma - \sigma_0)^3 + \dots, \quad (3)$$

where

$$\vec{T} = \frac{1}{2} \left(\vec{F}''(u_0) + \vec{G}''(v_0) \right), \quad \vec{R} = \frac{1}{2} \left(\vec{F}'''(u_0) - \vec{G}'''(v_0) \right). \quad (4)$$

If we assume the intersection to be regular the paths are not parallel at the intersection. Since $\vec{F}''(u_0) \neq \vec{0}$ and $\vec{G}''(v_0) \neq \vec{0}$ are the tangents to the paths at the intersection no linear combination of these vectors can vanish, and consequently $\vec{T}$ is non-zero. It follows from $\vec{F}'(u) \cdot \vec{F}'(u) = \vec{G}'(v) \cdot \vec{G}'(v) = 1$ that

$$\vec{F}'(u) \cdot \vec{F}''(u) = \vec{G}'(v) \cdot \vec{G}''(v) = 0 .$$

We then find that

$$\vec{F}'(u_0) \cdot \vec{T} = \frac{1}{2} \vec{F}'(u_0) \cdot \vec{F}''(u_0) + \frac{1}{2} \vec{G}'(v_0) \cdot \vec{G}''(v_0) = 0 ,$$

where we used the intersection condition $\vec{F}'(u_0) = \vec{G}'(v_0)$. We learn that the velocity of the cusp lies in the plane orthogonal to $\vec{T}$ at the cusp.

(b) We position the axes so that $\vec{T}$ lies along the positive y axis and $\vec{R}$ on the (x, y) plane, with components (R_x, R_y) . It follows from Eq. (3) that close enough to the cusp the string is contained in the (x, y) plane with position

$$x \simeq \frac{1}{6} R_x (\sigma - \sigma_0)^3 , \quad y \simeq \frac{1}{2} |\vec{T}| (\sigma - \sigma_0)^2 + \frac{1}{6} R_y (\sigma - \sigma_0)^3 . \quad (5)$$

For $\sigma - \sigma_0$ sufficiently small, the cubic term in y is negligible compared to the quadratic term. Eliminating $(\sigma - \sigma_0)$ from the relations $y \sim (\sigma - \sigma_0)^2$ and $x \sim (\sigma - \sigma_0)^3$ we find

$$y \sim x^{2/3} . \quad (6)$$

From part (a), we know that the velocity of the cusp lies in the plane orthogonal to $\vec{T}$, so it lies in the (x, z) plane.

(c) We are given that

$$\vec{F}(u) = \frac{\sigma_1}{2\pi} \left(\sin \frac{2\pi u}{\sigma_1} , -\cos \frac{2\pi u}{\sigma_1} , 0 \right) , \quad \vec{G}(v) = \frac{\sigma_1}{4\pi} \left(\sin \frac{4\pi v}{\sigma_1} , 0 , -\cos \frac{4\pi v}{\sigma_1} \right) . \quad (7)$$

Taking derivatives we find

$$\vec{F}'(u) = \left(\cos \frac{2\pi u}{\sigma_1} , \sin \frac{2\pi u}{\sigma_1} , 0 \right) , \quad \vec{G}'(v) = \left(\cos \frac{4\pi v}{\sigma_1} , 0 , \sin \frac{4\pi v}{\sigma_1} \right) .$$

We readily verify that $\vec{F}'(u) \cdot \vec{F}'(u) = \vec{G}'(v) \cdot \vec{G}'(v) = 1$ and that $\vec{F}'(u + \sigma_1) = \vec{F}'(u)$ as well as $\vec{G}'(v + \sigma_1) = \vec{G}'(v)$. At $t = \sigma = 0$, a short calculation using the results from (b) and (c) give

$$\frac{\partial \vec{X}}{\partial t}(0, 0) = c(1, 0, 0) , \quad \vec{T} = \frac{\pi}{\sigma_1} (0, 1, 2) , \quad \vec{R} = \frac{6\pi^2}{\sigma_1^2} (1, 0, 0) . \quad (8)$$

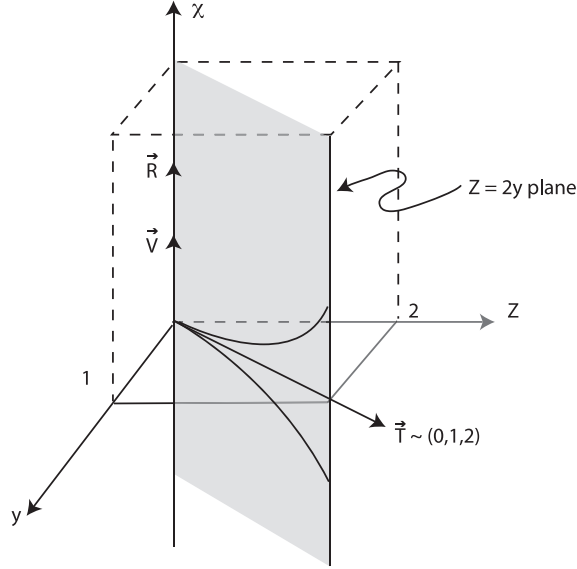


Figure 1: The cusp formed by the string at $ct\sigma = 0$. The cusp opens up in the direction of $\vec{T}$, it lies on the $z = 2y$ plane (shown shaded), and moves with the speed of light along the x axis.

The cusp is at the origin, opens up in the direction of $\vec{T} \sim (0, 1, 2)$, and lies on the plane spanned by $\vec{R}$ and $\vec{T}$, the $2y - z = 0$ plane. It moves with the speed of light in the direction of the positive x -axis, which happens to lie on the plane of the cusp. A sketch is shown in Figure 1.

(d) The function $\vec{F}(u)$ is explicitly periodic with period σ_1 , and the function $\vec{G}(v)$ is explicitly periodic with period $\sigma_1/2$. Using this information, let us write an expression for $\vec{X}\left(t + \frac{\sigma_1}{4c}, \sigma\right)$ and see if we can relate it to $\vec{X}(t, \sigma)$:

$$\vec{X}\left(t + \frac{\sigma_1}{4c}, \sigma\right) = \frac{1}{2} \left[\vec{F}\left(ct + \sigma + \frac{\sigma_1}{4}\right) + \vec{G}\left(ct + \frac{\sigma_1}{4} - \sigma\right) \right]. \quad (9)$$

Since σ is an arbitrary parameter with no meaning, we can define a shifted parameter

$$\tilde{\sigma} = \sigma + \frac{\sigma_1}{4}, \quad (10)$$

so that

$$\begin{aligned}
\vec{X}\left(t + \frac{\sigma_1}{4c}, \sigma\right) &= \frac{1}{2} \left[\vec{F}(ct + \tilde{\sigma}) + \vec{G}\left(ct - \tilde{\sigma} + \frac{\sigma_1}{2}\right) \right] \\
&= \frac{1}{2} \left[\vec{F}(ct + \tilde{\sigma}) + \vec{G}(ct - \tilde{\sigma}) \right] \quad (11) \\
\vec{X}\left(t + \frac{\sigma_1}{4c}, \sigma\right) &= \vec{X}(t, \tilde{\sigma}) = \vec{X}\left(t, \sigma + \frac{\sigma_1}{4}\right),
\end{aligned}$$

where in the second line we used the fact that $\vec{G}(v)$ is periodic with period $\sigma_1/2$. Thus, although the parameterization is not the same, it can still be said that as σ sweeps over its full range, $\vec{X}(t + \frac{\sigma_1}{4c}, \sigma)$ traces over the same string path as $\vec{X}(t, \sigma)$ did.

Comment: It is a general fact that for a closed string motion

$$\vec{X}(t, \sigma) = \frac{1}{2} \left(\vec{F}(ct + \sigma) + \vec{G}(ct - \sigma) \right)$$

the periodicity of **one** of the two functions, say $\vec{G}$, for example:

$$\vec{G}(u + \sigma_*) = \vec{G}(u)$$

guarantees that the curve traced by the string repeats itself in time every $\sigma_*/(2c)$. For this examine the string at a time $t + t_p$:

$$\vec{X}(t + t_p, \sigma) = \frac{1}{2} \left(\vec{F}(ct + ct_p + \sigma) + \vec{G}(ct + ct_p - \sigma) \right)$$

Letting

$$\tilde{\sigma} = \sigma + ct_p,$$

we find

$$\vec{X}(t + t_p, \sigma) = \frac{1}{2} \left(\vec{F}(ct + \tilde{\sigma}) + \vec{G}(ct + 2ct_p - \tilde{\sigma}) \right) \quad (A)$$

If we choose

$$2ct_p = \sigma_* \quad \rightarrow \quad t_p = \frac{\sigma_*}{2c} \quad \text{and} \quad \tilde{\sigma} = \sigma + \frac{\sigma_*}{2},$$

the $2ct_p$ in $\vec{G}$ can be ignored due to the periodicity and back in (A) we get

$$\vec{X}\left(t + \frac{\sigma_*}{2c}, \sigma\right) = \frac{1}{2} \left(\vec{F}(ct + \tilde{\sigma}) + \vec{G}(ct - \tilde{\sigma}) \right) = \vec{X}(t, \tilde{\sigma}).$$

We have thus shown that we get the same curve, with parameterization shifted:

$$\vec{X}\left(t + \frac{\sigma_*}{2c}, \sigma\right) = \vec{X}\left(t, \sigma + \frac{\sigma_*}{2}\right).$$

We now resume the solution of the problem. The condition for having a cusp is $\vec{F}'(u_0) = \vec{G}'(v_0)$ or, written out explicitly,

$$\cos \frac{2\pi u_0}{\sigma_1} = \cos \frac{4\pi v_0}{\sigma_1}, \quad \sin \frac{2\pi u_0}{\sigma_1} = 0, \quad \sin \frac{4\pi v_0}{\sigma_1} = 0. \quad (9)$$

The last two equations give

$$\frac{2\pi u_0}{\sigma_1} = m\pi, \quad \frac{4\pi v_0}{\sigma_1} = n\pi, \quad m, n \in \mathbb{Z}. \quad (10)$$

The first equation in (9) tells that the arguments of the cosine functions can differ by multiples of 2π . Using (12) we get

$$m\pi + 2\pi k = n\pi \quad \rightarrow \quad n = m + 2k, \quad k \in \mathbb{Z}. \quad (11)$$

It now follows from the last two equations that

$$\frac{u_0}{\sigma_1} = \frac{ct_0}{\sigma_1} + \frac{\sigma_0}{\sigma_1} = \frac{m}{2}, \quad \frac{v_0}{\sigma_1} = \frac{ct_0}{\sigma_1} - \frac{\sigma_0}{\sigma_1} = \frac{m}{4} + \frac{k}{2},$$

Solving for ct_0 and σ_0 we get

$$\frac{ct_0}{\sigma_1} = \frac{3m + 2k}{8}, \quad \frac{\sigma_0}{\sigma_1} = \frac{m - 2k}{8}. \quad (12)$$

Letting m and k run over all possible integers we get all the cusps in the time evolution of the closed string.

Two comments:

- Since m and k are arbitrary integers, so is $3m + 2k$. This means that during the period $ct_0/\sigma_1 \in [0, 1]$ we get cusps for $ct_0/\sigma_1 = q/8$ with $q = 0, 1, \dots, 7$.
- There is only one cusp at any of the above times. Proof: For any given t_0 the only ambiguity in the associated m and k is $m \rightarrow m + 2$ together with $k \rightarrow k - 3$. These changes make $\sigma_0 \rightarrow \sigma_0 + \sigma_1$, which is the same point.

The above two comments imply that we get eight cusps over the period.

7.8 Counting geodesics on a cone.

Consider the Figure 7.8 in the book (p.153). The shaded region is the cone, with angle α at the apex A . We will count the number of geodesics joining P and Q .

(a) Let us consider the copies of the cone that appear in the clockwise direction. Measuring angles clockwise with respect to the AP ray, the copies of Q appear at angles

$$\alpha + \phi, 2\alpha + \phi, 3\alpha + \phi, \dots$$

There are geodesics joining P to copies of Q as long as the Q copy does not go beyond the line obtained by extending the AP ray beyond the apex, namely, as long as its angle is less than or equal to π . The geodesics are simply the straight lines joining P to the Q copies and all go above the apex A in the figure. Using rotations by multiples of α those straight lines could be drawn, if desired, on the fundamental domain. Because of the above constraint, the last geodesic, call it the n -th one, is the one for which

$$n \text{ is the largest integer for which } n\alpha + \phi \leq \pi,$$

or equivalently

$$n \text{ is the largest integer for which } n \leq \frac{\pi - \phi}{\alpha}.$$

This is written as

$$n = \left\lfloor \frac{\pi - \phi}{\alpha} \right\rfloor, \tag{1}$$

where $[x]$ denotes the largest integer less than or equal to x . This derivation, strictly speaking is valid for $\alpha < \pi$. Since $\phi < \alpha$ we can then guarantee that n is not negative.

The copies of Q that develop in the clockwise direction but go beyond angle π (with respect to AP) do not give geodesics PQ because such lines go under the apex and thus travel on another set of copies of the cone. So we now consider the copies of the cone that are formed in the counterclockwise direction. Measuring angles in the counterclockwise direction with respect to AP we find copies of Q at angles

$$\alpha - \phi, 2\alpha - \phi, 3\alpha - \phi, \dots$$

Again, we find geodesics (going below the apex) from P to the above copies as long as the angle of the Q copy does not exceed π . The last geodesic, call it the m -th one, is the one for which

$$m \text{ is the largest integer for which } m\alpha - \phi \leq \pi,$$

or equivalently

$$m \text{ is the largest integer for which } m \leq \frac{\pi + \phi}{\alpha}.$$

This is written as

$$m = \left\lfloor \frac{\pi + \phi}{\alpha} \right\rfloor. \quad (2)$$

The total number N of geodesics is $n + m + 1$, where the ‘one’ counts the geodesic PQ in the original shaded domain (again for $\alpha < \pi$, which guarantees $\phi < \pi$). We have thus justified the formula

$$N(\alpha, \phi) = \left\lfloor \frac{\pi - \phi}{\alpha} \right\rfloor + \left\lfloor \frac{\pi + \phi}{\alpha} \right\rfloor + 1. \quad (3)$$

One can also verify that this formula holds for $\alpha \geq \pi$ (this takes a different picture to do it efficiently). In this case, however, it is possible for $[x]$ to take negative values.

(b) Consider the change $\phi \rightarrow \alpha - \phi$ in (3):

$$\begin{aligned} N(\alpha, \phi) \rightarrow N(\alpha, \alpha - \phi) &= \left\lfloor \frac{\pi - (\alpha - \phi)}{\alpha} \right\rfloor + \left\lfloor \frac{\pi + \alpha - \phi}{\alpha} \right\rfloor + 1. \\ &= \left\lfloor \frac{\pi + \phi}{\alpha} - 1 \right\rfloor + \left\lfloor \frac{\pi - \phi}{\alpha} + 1 \right\rfloor + 1. \\ &= \left\lfloor \frac{\pi + \phi}{\alpha} \right\rfloor - 1 + \left\lfloor \frac{\pi - \phi}{\alpha} \right\rfloor + 1 + 1. \\ &= \left\lfloor \frac{\pi - \phi}{\alpha} \right\rfloor + \left\lfloor \frac{\pi + \phi}{\alpha} \right\rfloor + 1 = N(\alpha, \phi). \end{aligned}$$

Indeed, $N(\alpha, \phi)$ is invariant under the indicated replacement. Note that we defined $\phi > 0$, measured clockwise from P to Q . Clearly $\phi < \alpha$, so $\alpha - \phi > 0$ is the angle measured clockwise from Q to P . Thus $N(\alpha, \alpha - \phi)$ counts the number of geodesics from Q to P . Since any geodesic from P to Q is also a geodesic from Q to P , the number of geodesics from P to Q is equal to the number of geodesics from Q to P . This means $N(\alpha, \phi) = N(\alpha, \alpha - \phi)$. Thus, we should have expected this invariance. Alternatively, we have performed a good consistency check.

(c) For $\Delta < \pi$ we have that the full angle at the apex is $\alpha > \pi$. It is then easier to do a direct analysis based on Figure 7.6 of the text, where we noted that for lensing to occur O must lie on the sector MAM' . We thus get

$$\text{Two geodesics if : } \pi - \Delta < \phi < \pi \rightarrow \alpha - \pi < \phi < \pi.$$

If this inequality is not satisfied we get one geodesic. Thus N takes values 1 or 2.

Problem 8.1: Lorentz invariants and their densities[†]

In the frame S , the box is at rest and the six walls lie in the planes $x = 0$, $x = L$, $y = 0$, $y = L$, $z = 0$, and $z = L$. In the frame S' , the box will appear Lorentz-contracted. That is, the box will be contracted in the direction of motion, the x -direction, by a factor of

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} . \quad (1)$$

Since the Lorentz contraction is not explicitly described in the textbook, it seems worthwhile to show how it can be found from the Lorentz transformations, Eqs. (2.34) in the textbook:

$$\begin{aligned} x' &= \gamma(x - vt) , & ct' &= \gamma\left(ct - \frac{vx}{c}\right) , \\ y' &= y , & z' &= z . \end{aligned} \quad (2)$$

It is a bit tricky, since we must remember that both t and x need to be transformed. The wall at $x = 0$ will appear in the S' frame to be moving, with

$$x'_1 = -\gamma vt , \quad t' = \gamma t , \quad \implies \quad x' = -vt' .$$

The wall at $x = L$ will also appear to be moving, with

$$x'_2 = \gamma(L - vt) , \quad t' = \gamma\left(t - \frac{vL}{c^2}\right) .$$

Solving the 2nd equation for t gives

$$t = \frac{t'}{\gamma} + \frac{vL}{c^2} ,$$

and then

$$x'_2 = \gamma\left(L - \frac{vt'}{\gamma} - \frac{v^2L}{c^2}\right) = -vt' + \frac{L}{\gamma} .$$

It follows that, as claimed, the distance between the walls in the S' frame is given by

$$x'_2 - x'_1 = \frac{L}{\gamma} . \quad (3)$$

(a) Since $y' = y$ and $z' = z$, the walls perpendicular to the y and z axes will lie in the planes $y' = 0$, $y' = L$, $z' = 0$, and $z' = L$. Thus, the lengths in these directions are measured in the S' frame as L , the same as in the rest frame S . The volume of the box in the S' frame, therefore, is

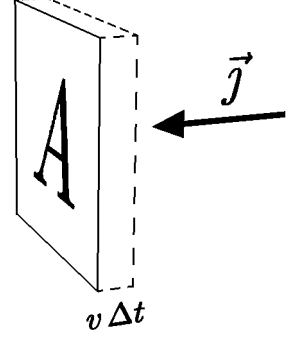
$$V' = \frac{L^3}{\gamma} = \frac{V}{\gamma} . \quad (4)$$

Since the total charge in the box is Lorentz-invariant, the charge density ρ' must be larger than in the rest frame by a factor of γ :

$$\boxed{\rho' = \gamma \rho_0} . \quad (5)$$

The current density inside the box can be found by considering a small rectangular area A , oriented perpendicular to the current flow. In a short time Δt , all the charge that was within the rectangular solid shown on the diagram will pass through the loop. The volume of this rectangular solid is $Av \Delta t$, so the charge passing through the loop is $\Delta Q = \rho' Av \Delta t$. The magnitude of the current density is then $j' = \Delta Q / (A \Delta t)$, so $j' = \rho' v$. As a vector equation, this can be written as

$$\boxed{\vec{j}' = \rho' \vec{v}'} , \quad (6)$$



where $\vec{v}' = -v\hat{z}$ is the velocity of the charge, as measured in the primed frame. Thus, the current density 4-vector inside the box can be written

$$j'^{\mu} = (c\rho', \vec{j}') = (c\gamma\rho_0, -\gamma\rho_0 v, 0, 0) . \quad (7)$$

In the rest frame S the current density 4-vector is given by

$$j^{\mu} = (c\rho_0, 0, 0, 0) , \quad (8)$$

so it can then be seen that the two 4-vectors are indeed related by the Lorentz transformation:

$$\begin{aligned} j'^0 &= \gamma(j^0 - \beta j^1) & \iff & c\gamma\rho_0 = \gamma(c\rho_0 - \beta \cdot 0) , \\ j'^1 &= \gamma(-\beta j^0 + j^1) & \iff & -\gamma\rho_0 v = \gamma(-\beta c\rho_0 + 0) , \\ j'^2 &= j^2 & \iff & 0 = 0 , \\ j'^3 &= j^3 & \iff & 0 = 0 . \end{aligned}$$

(b) Note that Eqs. (5) and (6) can be combined to give a general 4-vector equation for j^{μ} in terms of the rest frame charge density ρ_0 and the 4-velocity dx^{μ}/ds of the charge:

$$\boxed{j^{\mu}(x) = \rho_0(x) \frac{dx^{\mu}}{ds}} . \quad (9)$$

In the rest frame this reduces to $j^{\mu} = (\rho_0[d(ct)/ds], 0, 0, 0) = (c\rho_0, 0, 0, 0)$, and in S' one uses $dt'/ds = \gamma$ to see that it agrees with Eq. (7) above.

Problem 8.2. Lorentz invariance of electric charge.

In the unprimed frame S the total charge is given by

$$Q \equiv \int d^3x \rho(t=0, \vec{x}) , \quad (1)$$

and in the primed frame S' it is given by

$$Q' \equiv \int d^3x' \rho'(t'=0, \vec{x}') . \quad (2)$$

To prove that $Q' = Q$ we will have to use the conservation

$$\partial_\alpha j^\alpha = 0 \quad \leftrightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 , \quad j^\mu = (c\rho, \vec{j}) . \quad (3)$$

(a) The transformation equations that we will need are given by

$$\begin{aligned} t' &= \gamma \left(t - \frac{vx}{c^2} \right) & t &= \gamma \left(t' + \frac{vx'}{c^2} \right) & \rho' &= \gamma \left(\rho - \frac{vj^x}{c^2} \right) \\ x' &= \gamma(x - vt) & x &= \gamma(x' + vt') & j'^x &= \gamma(j^x - \rho v) \\ y' &= y & y &= y' & j'^y &= j^y \\ z' &= z & z &= z' & j'^z &= j^z . \end{aligned} \quad (4)$$

Using the transformation of ρ' , equation (2) gives us

$$Q' = \int d^3x' \gamma \left(\rho - \frac{vj^x}{c^2} \right) \Big|_{t'=0, \vec{x}'} .$$

The notation above is intended to mean that ρ and j^x are evaluated at the spacetime point described by $(t'=0, \vec{x}')$. However, ρ and j^x are defined as functions of the unprimed coordinates, so we need to write the arguments as the values of (t, x, y, z) that correspond to $(t'=0, \vec{x}')$. So, using the second column of transformations in (4) gives

$$Q' = \int d^3x' \gamma \left[\rho \left(\gamma \frac{vx'}{c^2}, \gamma x', y', z' \right) - \frac{v}{c^2} j^x \left(\gamma \frac{vx'}{c^2}, \gamma x', y', z' \right) \right] .$$

Using $x = \gamma x'$, $y = y'$, and $z = z'$ we find $\gamma d^3x' = d^3x$, so

$$Q'(v) = \int d^3x \left[\rho \left(\frac{vx}{c^2}, x, y, z \right) - \frac{v}{c^2} j^x \left(\frac{vx}{c^2}, x, y, z \right) \right] . \quad (5)$$

We have made explicit the apparent v dependence of Q' by writing $Q'(v)$. If $v = 0$ we see that the $\boxed{Q'(0) = Q}$, as given in (1). We need to show that $Q'(v)$ in fact does not depend on v .

(b) We now differentiate $Q'(v)$ in (5) with respect to v , using the chain rule to account for the fact that the value of the first argument (t) depends on v . One finds

$$\begin{aligned} \frac{dQ'}{dv} &= \int d^3x \left[\frac{x}{c^2} \frac{\partial \rho}{\partial t} - \frac{1}{c^2} j^x - \frac{vx}{c^4} \frac{\partial j^x}{\partial t} \right] \Big|_{t=\frac{vx}{c^2}, x, y, z} \\ &= \frac{1}{c^2} \int d^3x \left[x \left[\frac{\partial \rho}{\partial t} - \frac{v}{c^2} \frac{\partial j^x}{\partial t} \right] - j^x \right] \Big|_{t=\frac{vx}{c^2}, x, y, z} . \end{aligned} \quad (6)$$

Here the notation indicates that the right-hand side is evaluated at $(t, x, y, z) = (\frac{vx}{c^2}, x, y, z)$, but only **after** the derivatives are calculated.

(c) Now we want to use Eq. (3) to replace $\partial \rho / \partial t$, but we need to be careful. To define partial derivatives, such as those appearing in Eq. (3), one must specify what variables are being held fixed. The partial derivatives of Eq. (3) are defined by the default prescription for spacetime functions: the independent variables are taken as t, x, y , and z , and any partial derivative with respect to one of them is taken with the other three independent variables fixed. Evaluating Eq. (3) at the coordinates needed for Eq. (6), we can write

$$\frac{\partial \rho}{\partial t} \Big|_{t=\frac{vx}{c^2}, x, y, z} = - \left[\frac{\partial j^x}{\partial x} + \frac{\partial j^y}{\partial y} + \frac{\partial j^z}{\partial z} \right] \Big|_{t=\frac{vx}{c^2}, x, y, z} . \quad (7)$$

In the integrand of Eq. (6), however, t is not an independent variable, but is determined by $t = vx/c^2$. For this reason it will be useful to introduce partial derivatives appropriate to the independent variable set of x, y , and z only. In particular, we can write

$$\left(\frac{\partial}{\partial x} j^x \left(\frac{vx}{c^2}, x, y, z \right) \right)_{y, z} = \left[\frac{v}{c^2} \frac{\partial j^x}{\partial t} + \frac{\partial j^x}{\partial x} \right] \Big|_{t=\frac{vx}{c^2}, x, y, z} , \quad (8)$$

where the subscript y, z on the left-hand side indicates that the partial derivative is taken with these variables (and not t) held fixed. The symbols on the right-hand side have the same meaning as in previous equations. We wish to eliminate $\partial j^x / \partial x$ from Eqs. (7) and (8), which can be done by adding them to give

$$\left[\frac{\partial \rho}{\partial t} - \frac{v}{c^2} \frac{\partial j^x}{\partial t} \right] \Big|_{t=\frac{vx}{c^2}, x, y, z} = - \left(\frac{\partial}{\partial x} j^x \left(\frac{vx}{c^2}, x, y, z \right) \right)_{y, z} - \left[\frac{\partial j^y}{\partial y} + \frac{\partial j^z}{\partial z} \right] \Big|_{t=\frac{vx}{c^2}, x, y, z} . \quad (9)$$

Substituting Eq. (9) into Eq. (6), one finds that

$$\frac{dQ'}{dv} = -\frac{1}{c^2} \int d^3x \left\{ x \left(\frac{\partial}{\partial x} j^x \left(\frac{vx}{c^2}, x, y, z \right) \right)_{y, z} + \left[x \left(\frac{\partial j^y}{\partial y} + \frac{\partial j^z}{\partial z} \right) + j^x \right] \Big|_{t=\frac{vx}{c^2}, x, y, z} \right\} . \quad (10)$$

Let us consider the term on the right-hand side proportional to $\partial j^y / \partial y$, which we can write as

$$-\frac{1}{c^2} \int d^3x \left(\frac{\partial}{\partial y} \left[x j^y \left(\frac{vx}{c^2}, x, y, z \right) \right] \right)_{x,z} . \quad (11)$$

The integral over y just gives boundary terms evaluated at $y = \pm\infty$. These terms vanish since we assume that the charge and current densities are localized and hence vanish at infinity. Similarly, the term proportional to $\partial j^z / \partial z$ vanishes when integrated over z . The remaining terms in (10) are

$$\frac{dQ'}{dv} = -\frac{1}{c^2} \int d^3x \left\{ x \left(\frac{\partial}{\partial x} j^x \left(\frac{vx}{c^2}, x, y, z \right) \right)_{y,z} + j^x|_{t=\frac{vx}{c^2}, x, y, z} \right\} . \quad (12)$$

The two terms combine to give a total derivative, which vanishes when integrated over x :

$$\frac{dQ'}{dv} = -\frac{1}{c^2} \int d^3x \left(\frac{\partial}{\partial x} \left[x j^x \left(\frac{vx}{c^2}, x, y, z \right) \right] \right)_{y,z} = 0 . \quad (13)$$

Thus,

$$\frac{dQ'(v)}{dv} = 0, \quad \text{for all } v > 0 . \quad (14)$$

It then follows that

$$Q'(v) = Q'(0) + \int_0^v du \frac{dQ'(u)}{du} = Q'(0) = Q . \quad (15)$$

We have thus shown that

$Q'(v) = Q ,$

(16)

which is the result we were trying to show.

[†]Solution written by Alan Guth (edits by B. Zwiebach).

8.1 Angular momentum as conserved charge

a) $\delta \vec{q}(t) = \epsilon \vec{n} \times \vec{q}(t)$

represents a rotation by angle ϵ about the unit vector $\vec{n}$.

$$\delta \dot{\vec{q}} = \epsilon \vec{n} \times \dot{\vec{q}} \quad \text{means} \quad \delta |\dot{\vec{q}}|^2 = 2 \dot{\vec{q}} \cdot \delta \dot{\vec{q}} = 0$$

so if $L = L(|\dot{\vec{q}}|)$, $\delta L = 0$

b) Conserved quantity $Q(\vec{n}) = \frac{1}{\epsilon} \frac{\partial L}{\partial \dot{\vec{q}}} \cdot \delta \dot{\vec{q}}$

$$= \vec{p} \cdot \vec{n} \times \vec{q} = \vec{n} \cdot \vec{q} \times \vec{p}$$

where $\vec{p}$ is the canonical momentum $\frac{\partial L}{\partial \dot{\vec{q}}}$.

Since this is conserved for any $\vec{n}$, the vector $\vec{q} \times \vec{p}$ is conserved.

8.4 A generalization for charges and a special case for currents.

(a) With several coordinates q^a and symmetry parameters ϵ^i , the variations are written as

$$\delta q^a = \epsilon^i h_i^a(q; t). \quad (1)$$

We now claim that the charges are

$$\epsilon^i Q_i = \frac{\partial L}{\partial \dot{q}^a} \delta q^a. \quad (2)$$

This time the Euler-Lagrange and invariance equations are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^a} \right) - \frac{\partial L}{\partial q^a} = 0, \quad \frac{\partial L}{\partial q^a} \delta q^a + \frac{\partial L}{\partial \dot{q}^a} \frac{d}{dt} (\delta q^a) = 0. \quad (3)$$

We verify conservation as in the book: we compute the time derivative of the charges in (2) and then use equations (3) to find that

$$\epsilon^i \frac{dQ_i}{dt} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^a} \right) \delta q^a + \frac{\partial L}{\partial \dot{q}^a} \frac{d}{dt} (\delta q^a) = \frac{\partial L}{\partial q^a} \delta q^a + \frac{\partial L}{\partial \dot{q}^a} \frac{d}{dt} (\delta q^a) = 0. \quad (4)$$

Since the ϵ^i are independent parameters, we conclude that

$$\frac{dQ_i}{dt} = 0. \quad (5)$$

(b) The index α can only take the value zero – there is just ξ^0 , which plays the role of time in this world. The field ϕ^a is only a function of time. The action becomes

$$S = \int d\xi^0 \mathcal{L}(\phi^a, \partial_0 \phi^a). \quad (6)$$

The quoted equations become

$$\epsilon^i j_i^0 \equiv \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi^a)} \delta \phi^a, \quad \partial_0 j_i^0 = 0, \quad Q_i = j_i^0, \quad (7)$$

where the last equation is “obtained” because there are no spatial coordinates to integrate over; since there is no space, the charge density j_i^0 is really just a charge. The second equation in (7) is charge conservation, in correspondence with (5). The first equation in (7), takes the form of (2), once we note that ∂_0 is really a dot derivative, and that q^a and ϕ^a are the labels of the corresponding dynamical variables, both of which depend only on time.

8.3 Lorentz charges for the relativistic point particle

$$S = -mc \int d\tau \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \quad \dot{x}^\mu = \frac{dx^\mu}{d\tau}$$

We will treat all the coordinates x^μ like the q in section 8.2 or the X^μ of 8.3 with τ playing the role of time. Or we can think of this as like equation (8.17) with $\xi^0 = \tau$ the only "world" coordinate and x^μ playing the role of the ϕ^a .

a) With $\delta x^\mu = \epsilon^\mu$, $\delta \dot{x}^\mu = 0$ so $\delta L = 0$

The conserved charges are given by $\epsilon^\mu Q_\mu = \frac{\partial L}{\partial \dot{x}^\mu} \epsilon^\mu$

$\frac{\partial L}{\partial \dot{x}^\mu}$ is of course the canonical p_μ so $Q_\mu = p_\mu$

In fact $p_\mu = \frac{mc \eta_{\mu\nu} \dot{x}^\nu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} = mc \frac{dx_\mu}{dS}$ which is of

course constant for a free particle

b) Infinitesimal Lorentz transformation is $\delta x^\mu = \epsilon^{\mu\nu} x_\nu$

with $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ (as in (8.49))

This is defined so that it leaves $\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ invariant.

c) Conserved charge associated with $\epsilon^{\mu\nu}$ is $\epsilon^{\mu\nu} Q_{\mu\nu} = \frac{\partial L}{\partial \dot{x}^\mu} \epsilon^{\mu\nu} x_\nu$

where $Q_{\mu\nu} = -Q_{\nu\mu}$ (the symmetric part of $Q_{\mu\nu}$ could be anything and would not be conserved)

Hence
$$Q_{\mu\nu} = \frac{1}{2} \left(x_\nu \frac{\partial L}{\partial \dot{x}_\mu} - x_\mu \frac{\partial L}{\partial \dot{x}_\nu} \right)$$

$$= \frac{1}{2} (x_\nu p_\mu - x_\mu p_\nu)$$

If we want Q_{ij} to correspond to angular momentum we should have $Q_{ij} = x_i p_j - x_j p_i$ for i, j spatial

so we take $Q_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$

$$\frac{dQ_{\mu\nu}}{dt} = (\dot{x}_\mu p_\nu - \dot{x}_\nu p_\mu) + (x_\mu \dot{p}_\nu - x_\nu \dot{p}_\mu)$$

Since $\dot{p}_\mu = 0$ (equation of motion) and $p_\mu = f(t) \dot{x}_\mu$ we verify that $Q_{\mu\nu}$ is constant.

8.6 Simple estimates regarding α' , T_0 , and ℓ_s .

We know that

$$T_0 = \frac{1}{2\pi\alpha' \hbar c} \quad \text{and} \quad \ell_s = \hbar c \sqrt{\alpha'}.$$

Moreover, for quick estimates we can use

$$\hbar c = 198 \text{ MeV} \times 10^{-15} \text{ m} = 1.98 \times 10^{-14} \text{ GeV} \cdot \text{cm},$$

together with

$$\frac{\text{GeV}}{\text{cm}} = 1.6 \times 10^{-8} \text{ N} \simeq 1.63 \times 10^{-12} \text{ ton}.$$

(a) For $\alpha' \simeq 0.95 \text{ GeV}^{-2}$, we get

$$T_0 = \frac{1}{2 \cdot 3.14 \cdot 0.95 \cdot 1.98 \times 10^{-14}} \frac{\text{GeV}}{\text{cm}} \simeq 8.47 \times 10^{12} \frac{\text{GeV}}{\text{cm}} \simeq 13.8 \text{ ton}.$$

The string length is then

$$\ell_s = 1.98 \times 10^{-14} \times \sqrt{0.95} \text{ cm} = 1.93 \times 10^{-14} \text{ cm}.$$

(b) With $\ell_s \sim 10^{-30} \text{ cm}$, we have

$$\alpha' = \left(\frac{\ell_s}{\hbar c} \right)^2 = \frac{10^{-60}}{3.92 \times 10^{-28}} \text{ GeV}^{-2} = 2.55 \times 10^{-33} \text{ GeV}^{-2}.$$

Since the string tension is inversely proportional to α' we have

$$T_0 = 13.8 \text{ ton} \times \left(\frac{0.95 \text{ GeV}^{-2}}{2.55 \times 10^{-33} \text{ GeV}^{-2}} \right) = 5.14 \times 10^{33} \text{ ton}.$$

This is a fantastically large force. The gravitational force which keeps the Earth in orbit around the sun is about $3.6 \times 10^{18} \text{ ton}$. The above string tension is about 2.6 million times larger than the “weight” of the sun (the mass of the sun is $\simeq 2 \times 10^{30} \text{ kg}$).

8.7 Angular momentum of a rotating string.

We are given that

$$d\vec{p} = \frac{T_0 ds}{c^2} \frac{\vec{v}_\perp}{\sqrt{1 - \frac{v_\perp^2}{c^2}}}.$$

If we parameterize half a string, from the center to its edge, with $s \in [0, \ell/2]$ we note that

$$|\vec{v}_\perp(s)| = \frac{2s}{\ell} c.$$

This is correct because, with constant angular velocity, the speed of the string increases linearly along the string: it is zero at the center $s = 0$ and reaches c at the endpoint $s = \ell/2$. The magnitude $dp = |d\vec{p}|$ of the little angular momentum carried by a piece of string of length ds at s is then:

$$dp = \frac{2T_0 s ds}{\ell c} \frac{1}{\sqrt{1 - \frac{4s^2}{\ell^2}}}.$$

The infinitesimal angular momentum dJ carried by the little piece of string is

$$dJ = |\vec{r} \times d\vec{p}| = s dp = \frac{2T_0 s^2 ds}{\ell c} \frac{1}{\sqrt{1 - \frac{4s^2}{\ell^2}}}.$$

Including the contribution from both string halves, the total angular momentum is

$$J = 2 \int_0^{\ell/2} \frac{2T_0 s^2 ds}{\ell c} \frac{1}{\sqrt{1 - \frac{4s^2}{\ell^2}}} = \frac{4T_0}{\ell c} \int_0^{\ell/2} \frac{s^2 ds}{\sqrt{1 - \frac{4s^2}{\ell^2}}}.$$

Letting $s = \frac{\ell}{2}x$ we get

$$J = \frac{4T_0 \ell^3}{\ell c} \frac{1}{8} \int_0^1 \frac{x^2 dx}{\sqrt{1 - x^2}}.$$

The integral has value $\pi/4$ and we thus obtain,

$$J = \frac{4T_0 \ell^3}{\ell c} \frac{\pi}{8 \cdot 4} = \frac{\pi T_0 \ell^2}{8c}.$$

With little rearrangement and recalling (7.60) we find

$$J = \frac{\pi^2 T_0^2 \ell^2}{4 \cdot 2\pi T_0 c} = \frac{1}{2\pi T_0 c} \cdot \left(\frac{1}{2}\pi T_0 \ell\right)^2 = \frac{1}{2\pi T_0 c} E^2,$$

in agreement with (8.75).

8.8 Angular momentum of Kasey's jumping rope.

From the solution to the jumping rope problem we have

$$(X_1, X_2) = \frac{L_0}{\pi} \tan \gamma \sin \frac{\pi \sigma}{\sigma_1} (\cos \omega t, \sin \omega t), \quad (1)$$

and, moreover,

$$\omega = \frac{\pi c}{L_0} \cos \gamma, \quad \sigma_1 = \frac{E}{T_0} = \frac{L_0}{\cos \gamma}. \quad (2)$$

The momentum density is computed using (1) and (2)

$$\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2} \frac{\partial \vec{X}}{\partial t} \quad \rightarrow \quad (\mathcal{P}_1^\tau, \mathcal{P}_2^\tau) = \frac{T_0}{c} \sin \gamma \sin \frac{\pi \sigma}{\sigma_1} (-\sin \omega t, \cos \omega t) \quad (3)$$

We compute the z -component J_z of the angular momentum of the string, measured with respect to the origin:

$$J_z = M_{12} = \int_0^{\sigma_1} d\sigma (X_1 \mathcal{P}_2^\tau - X_2 \mathcal{P}_1^\tau). \quad (4)$$

Using (1) and (3) we find

$$J_z = \frac{L_0}{\pi} \tan \gamma \frac{T_0}{c} \sin \gamma \int_0^{\sigma_1} d\sigma \sin^2 \frac{\pi \sigma}{\sigma_1} = \frac{L_0 T_0 \sigma_1}{2\pi c} \frac{\sin^2 \gamma}{\cos \gamma} = \frac{T_0}{2\pi c} \left(\frac{L_0}{\cos \gamma} \right)^2 \sin^2 \gamma. \quad (5)$$

We express this result in terms of energy using (2):

$$J_z = \frac{T_0}{2\pi c} \frac{E^2}{T_0^2} \sin^2 \gamma.$$

Equivalently,

$$\frac{J_z}{\hbar} = \frac{1}{2\pi T_0 \hbar c} E^2 \sin^2 \gamma = \sin^2 \gamma \alpha' E^2. \quad (6)$$

The above $J_z/\hbar$ is smaller than $\alpha' E^2$, the value of the angular momentum of a rigidly rotating straight open string of the same energy.

Problem 8.6 Generalizing the construction of conserved charges

When $q \rightarrow q + \delta q$, $\delta L = \frac{d}{dt}(\epsilon \dot{L})$

But $\delta L = \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q$ and the equation of motion

is $\frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}$ so $\delta L = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right)$

Hence $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q - \epsilon \dot{L} \right) = 0$ so that if we define Q by

$\epsilon Q = \frac{\partial L}{\partial \dot{q}} \delta q - \epsilon \dot{L}$ (remember $\delta q = \epsilon \dot{L}$)
 Q is a constant of the motion

With $\delta q = \epsilon \dot{q}$, $\delta L = \frac{\partial L}{\partial q} \epsilon \dot{q} + \frac{\partial L}{\partial \dot{q}} \epsilon \ddot{q} = \epsilon \frac{dL}{dt}$ so $\dot{L} = L$

(Note that at this stage we do not use the equation of motion. We have used $\frac{\partial L}{\partial t} = 0$. Note also that

$q + \delta q = q + \epsilon \dot{q} = q(t + \epsilon)$ so the symmetry is time translation)

$\epsilon Q = \frac{\partial L}{\partial \dot{q}} \epsilon \dot{q} - \epsilon L$ so $Q = \dot{q} \frac{\partial L}{\partial \dot{q}} - L = H$

So this construction tells us that when time translation is a symmetry (i.e. $\frac{\partial L}{\partial t} = 0$), the energy H is conserved.

8.10 Generalizing the construction of conserved currents.

To show that the current is conserved we take its divergence

$$\epsilon^i \partial_\alpha j_i^\alpha = \left(\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \right) \delta \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} (\partial_\alpha \delta \phi^a) - \epsilon^i \partial_\alpha \Lambda_i^\alpha \quad (1)$$

We now use the equations of motion that follow from the Lagrangian

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} = \frac{\partial \mathcal{L}}{\partial \phi^a}$$

to simplify the first term of (1). We then find

$$\epsilon^i \partial_\alpha j_i^\alpha = \frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} (\partial_\alpha \delta \phi^a) - \epsilon^i \partial_\alpha \Lambda_i^\alpha.$$

The first two terms on the right-hand side are simply the variation $\delta \mathcal{L}$ of the Lagrangian under the symmetry transformation. By equation (1) of the problem statement we have

$$\epsilon^i \partial_\alpha j_i^\alpha = \delta \mathcal{L} - \epsilon^i \partial_\alpha \Lambda_i^\alpha = 0.$$

Since the parameters ϵ^i are independent constants, the currents j_i are conserved.

We now consider a $\mathcal{L}$ with no dependence on the coordinates ξ^β . Under a change $\phi^a \rightarrow \phi^a + \epsilon^\beta \partial_\beta \phi^a$, the variation of the Lagrangian is given by

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi^a} \epsilon^\beta \partial_\beta \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \epsilon^\beta \partial_\alpha \partial_\beta \phi^a = \epsilon^\beta \left(\frac{\partial \mathcal{L}}{\partial \phi^a} \partial_\beta \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \partial_\beta (\partial_\alpha \phi^a) \right).$$

We therefore recognize that

$$\delta \mathcal{L} = \epsilon^\beta \frac{\partial \mathcal{L}}{\partial \xi^\beta} = \frac{\partial}{\partial \xi^\beta} (\epsilon^\beta \mathcal{L}),$$

where we also used the constancy of the ϵ^β . Relabelling indices

$$\delta \mathcal{L} = \frac{\partial}{\partial \xi^\alpha} (\epsilon^\alpha \mathcal{L}) = \frac{\partial}{\partial \xi^\alpha} (\epsilon^\beta \delta_\beta^\alpha \mathcal{L}) = \frac{\partial}{\partial \xi^\alpha} (\epsilon^\beta \Lambda_\beta^\alpha) \quad \text{with} \quad \Lambda_\beta^\alpha = \delta_\beta^\alpha \mathcal{L}.$$

We then have, using equation (2) of the problem statement

$$\epsilon^\beta j_\beta^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \epsilon^\beta \partial_\beta \phi^a - \epsilon^\beta \delta_\beta^\alpha \mathcal{L} \quad \rightarrow \quad j_\beta^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \partial_\beta \phi^a - \delta_\beta^\alpha \mathcal{L},$$

as desired. For $\alpha = \beta = 0$

$$j_0^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^a)} \partial_0 \phi^a - \mathcal{L} = \mathcal{H}.$$

where $\mathcal{H}$ is the Hamiltonian density of the system. This is clear because the first term on the right-hand side is the product of the canonical momentum associated with the coordinate ϕ^a times the velocity of the coordinate ϕ^a .

Problem 9.1 Vectors orthogonal to null vectors

$\eta_\mu n^\mu = 0 \Rightarrow n^\mu = (n^0, \vec{n})$ (where $\vec{n}$ is a $(D-1)$ dimensional vector) with $(n^0)^2 = |\vec{n}|^2$ i.e. $n^0 = \pm |\vec{n}|$

The space of vectors b^μ with $\eta_\mu b^\mu = 0$ is unchanged if we multiply n^μ by a constant, so we can take $|\vec{n}| = 1$, $n^0 = 1$, without loss of generality.

a) Let $b^\mu = (b^0, \vec{b})$, $\eta_\mu b^\mu = b^0 - \vec{n} \cdot \vec{b}$

$$\text{so } \eta_\mu b^\mu = 0 \Leftrightarrow b^0 = \vec{n} \cdot \vec{b}$$

$$\text{With } |\vec{n}| = 1, \quad |\vec{n} \cdot \vec{b}| \leq |\vec{b}| \quad \text{so } |b^0| \leq |\vec{b}|$$

which means b^μ is space-like ($|b^0| < |\vec{b}|$) or null ($|b^0| = |\vec{b}|$)

b) b^μ is only null if $|b^0| = |\vec{b}| = |\vec{n} \cdot \vec{b}|$ which is only if $\vec{b}$ is parallel or antiparallel to $\vec{n}$ i.e. if $\vec{b} = \lambda \vec{n}$ and then $b^0 = \vec{n} \cdot \vec{b} = \lambda$

$$\text{so } b^\mu = \lambda(1, \vec{n}) = \lambda n^\mu$$

c) $\vec{b}$ can be any $(D-1)$ dimensional vector so we can write $\vec{b} = \lambda_1 \vec{n} + \lambda_2 \vec{e}_2 + \dots + \lambda_{D-1} \vec{e}_{D-1}$ where

$\vec{e}_2, \dots, \vec{e}_{D-1}$ are $D-2$ linear independent vectors with $\vec{e}_i \cdot \vec{n} = 0$. In this case $b^0 = \vec{n} \cdot \vec{b} = \lambda_1$

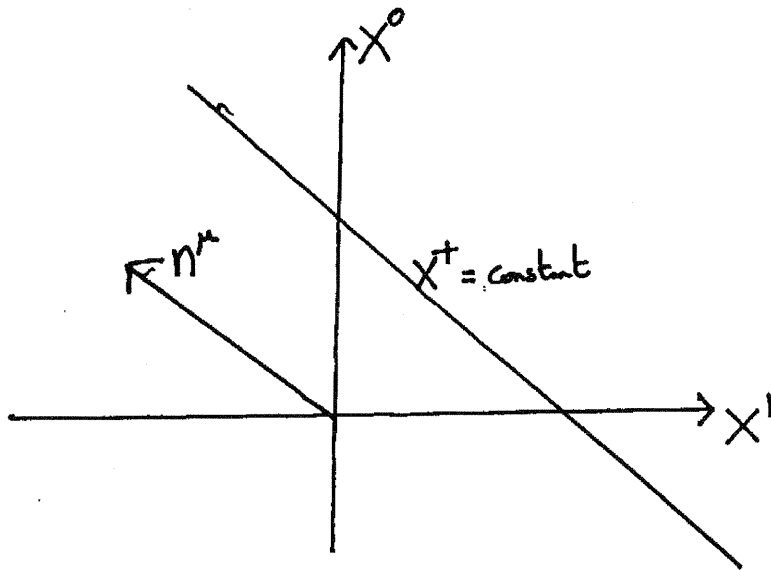
$$\text{Hence } b^\mu = \lambda_1 n^\mu + \lambda_2 (0, \vec{e}_2) + \dots + \lambda_{D-1} (0, \vec{e}_{D-1})$$

which shows that the b^μ lie in a $(D-1)$ dimensional linear subspace of D dimensional Minkowski space.

From (b), the null vectors b^μ are λn^μ i.e. here $\lambda_2 = \dots = \lambda_{D-1} = 0$ and lie in a 1 dimensional linear subspace of V .

[This is consistent. $n^\mu \in V$ since $n^\mu \eta_\mu = 0$. Also for $D=2$, take $n^\mu = (1, 1)$. V consists of (b, b) , all null vectors parallel to n^μ]

d) Since $X^+ = \frac{X^0 + X^1}{\sqrt{2}}$, we have $X^+ = \eta_\mu X^\mu$ with $\eta^\mu = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, \dots, 0)$



Problem 9.2 Consistency checks on the solution for X^-

a) From (9.66)
$$\frac{\partial X^-}{\partial \tau} \pm \frac{\partial X^-}{\partial \sigma} = \frac{1}{2} a \left(\frac{\partial X^I}{\partial \tau} \pm \frac{\partial X^I}{\partial \sigma} \right)^2$$

so
$$\frac{\partial X^-}{\partial \tau} = \frac{1}{2} a \left(\frac{\partial X^I}{\partial \tau} \right)^2 + \frac{1}{2} a \left(\frac{\partial X^I}{\partial \sigma} \right)^2 \quad \dots (3.1)$$

$$\frac{\partial X^-}{\partial \sigma} = a \frac{\partial X^I}{\partial \tau} \frac{\partial X^I}{\partial \sigma} \quad \dots (3.2)$$

[We should first check that (3.1) is consistent with (3.2),

i.e. that
$$\frac{\partial}{\partial \sigma} \left(\frac{\partial X^-}{\partial \tau} \right) = \frac{\partial}{\partial \tau} \left(\frac{\partial X^-}{\partial \sigma} \right)$$

From (3.1)
$$\frac{\partial}{\partial \sigma} \left(\frac{\partial X^-}{\partial \tau} \right) = a \frac{\partial X^I}{\partial \tau} \frac{\partial^2 X^I}{\partial \tau^2} + a \frac{\partial X^I}{\partial \sigma} \frac{\partial^2 X^I}{\partial \sigma^2}$$

and from (3.2)
$$\frac{\partial}{\partial \tau} \left(\frac{\partial X^-}{\partial \sigma} \right) = a \frac{\partial X^I}{\partial \tau} \frac{\partial^2 X^I}{\partial \tau^2} + a \frac{\partial X^I}{\partial \sigma} \frac{\partial^2 X^I}{\partial \sigma^2}$$

so these are equal provided
$$\frac{\partial^2 X^I}{\partial \sigma^2} = \frac{\partial^2 X^I}{\partial \tau^2}$$

b) From (3.1)
$$\frac{\partial^2 X^-}{\partial \tau^2} = a \frac{\partial X^I}{\partial \tau} \frac{\partial^2 X^I}{\partial \tau^2} + a \frac{\partial X^I}{\partial \sigma} \frac{\partial^2 X^I}{\partial \sigma^2}$$

and from (3.2)
$$\frac{\partial^2 X^-}{\partial \sigma^2} = a \frac{\partial X^I}{\partial \tau} \frac{\partial^2 X^I}{\partial \tau^2} + a \frac{\partial X^I}{\partial \sigma} \frac{\partial^2 X^I}{\partial \sigma^2}$$

so
$$\frac{\partial^2 X^-}{\partial \tau^2} = \frac{\partial^2 X^-}{\partial \sigma^2} \quad \text{provided} \quad \frac{\partial^2 X^I}{\partial \tau^2} = \frac{\partial^2 X^I}{\partial \sigma^2}$$

c) At endpoints, for each I either $\frac{\partial X^I}{\partial \sigma} = 0$ (Neumann)

or $X^I = \text{constant}$, i.e. $\frac{\partial X^I}{\partial \tau} = 0$ (Dirichlet)

So from (3.2), $\frac{\partial X^-}{\partial \sigma} = 0$ (Neumann)

Problem 9.3 Rotating open string in light-cone gauge

$$\alpha_1^I = (a, ia, 0, \dots) \quad \alpha_{-1}^I = (a, -ia, 0, \dots)$$

$(I=2,3,\dots)$ All other $\alpha_n^I = 0$

a) [Mass was intended to be M determined by (9.82),
 $M^2 = 2p^+p^- - p^I{}^2$. If our solution has $p^I = 0$
 and $p^+ = p^-$ so that $p^I = 0$, then $M = p^0$]
 From (9.82) $M^2 = \frac{1}{\alpha'} \sum_I |\alpha_1^I|^2 = \frac{2a^2}{\alpha'}$, $M = \frac{2a}{\sqrt{2\alpha'}}$

b) From (9.68) $X^{(2)} = i\sqrt{2\alpha'} \left\{ \alpha_1^{(2)} e^{-i\tau} - \alpha_{-1}^{(2)} e^{i\tau} \right\} \cos\sigma$
 $= 2\sqrt{2\alpha'} a \cos\sigma \sin\tau$
 $X^{(3)} = i\sqrt{2\alpha'} \left\{ \alpha_1^{(3)} e^{-i\tau} - \alpha_{-1}^{(3)} e^{i\tau} \right\} \cos\sigma$
 $= -2\sqrt{2\alpha'} a \cos\sigma \cos\tau$

So at fixed τ , with $0 \leq \sigma \leq \pi$, string in (2,3) plane has
 length $\sqrt{2\alpha'} 4a$. We are going to arrange that $X' = 0$
 so that $\tau = \frac{1}{2\alpha' p^+} X^+ = \frac{1}{2\alpha' p^+} \frac{X^0}{\sqrt{2}}$ and string is rotating
 in (2,3) plane with angular velocity $\omega = \frac{1}{2\sqrt{2}\alpha' p^+}$

c) $L_n^\perp = \frac{1}{2} \sum_{p \neq 0} \alpha_{n-p}^I \alpha_p^I$ (note: no complex conjugation)
 Since only $\alpha_{\pm 1}$ are non-zero, only $L_0, L_{\pm 2}$ can be non-zero.
 $L_{\pm 2}^\perp = \frac{1}{2} \alpha_{\pm 1}^I \alpha_{\pm 1}^I = \frac{1}{2} \{ a^2 + (\pm ia)^2 \} = 0$
 $L_0^\perp = \frac{1}{2} (\alpha_1^I \alpha_{-1}^I + \alpha_{-1}^I \alpha_1^I) = \alpha_1^I \alpha_1^{I*} = 2a^2$

Hence from (9.79), with $x_0^- = 0$,

$$X^- = \frac{1}{p^+} 2a^2 \tau$$

d) From (9.63) $X^+ = 2\alpha' p^+ \tau$

To have $X' = 0$ we need $X^+ = X^-$

i.e. $\frac{2a^2}{p^+} = 2\alpha' p^+$, $p^+ = \frac{a}{\sqrt{\alpha'}}$

Then $X^\pm = \frac{X^0}{\sqrt{2}}$ so $X^0 = \sqrt{2} 2\alpha' \frac{a}{\sqrt{\alpha'}} \tau = 2a\sqrt{2\alpha'} \tau$

e) From the result at the end of (b), $\omega = \frac{1}{2\sqrt{2}\alpha'} \frac{\sqrt{\alpha'}}{a} = \frac{1}{\sqrt{2\alpha'}} \frac{1}{2a}$

whereas from (a) $M = \frac{2a}{\sqrt{2\alpha'}}$ and from (b) $L = \sqrt{2\alpha'} 4a$

so $\omega L = 2$ and $\frac{M}{L} = \frac{1}{4\alpha'} = \frac{\pi T_0}{2}$

which agrees with (7.66)

Problem 9.4 Generating constant open string motion

$$\alpha_1^I = \alpha_{-1}^I = \begin{pmatrix} 2 & 3 & 4 & \dots \\ a, 0, 0, \dots \end{pmatrix} \quad \text{All other } \alpha_n^I = 0$$

a) From (9.68) $X^{(2)} = i\sqrt{2\alpha'} \cos \sigma a (e^{-i\tau} - e^{i\tau})$
 $= \sqrt{2\alpha'} 2a \cos \sigma \sin \tau$

From (9.69) $X^+ = 2\alpha' p^+ \tau$

For X^- we need to calculate $L_n^\perp$ from (9.76)

Only $L_0^\perp, L_{\pm 2}^\perp$ are non-zero.

$$L_{\pm 2}^\perp = \frac{1}{2} \alpha_{\pm 1}^I \alpha_{\pm 1}^I = \frac{1}{2} a^2$$

$$L_0^\perp = \frac{1}{2} (\alpha_1^I \alpha_{-1}^I + \alpha_{-1}^I \alpha_1^I) = a^2$$

So from (9.71) with $x_0^- = 0$

$$X^- = \frac{a^2}{p^+} \tau + \frac{i}{p^+} \frac{\cos 2\sigma}{2} \frac{1}{2} a^2 (e^{-2i\tau} - e^{2i\tau})$$

$$= \frac{a^2}{p^+} \left\{ \tau + \frac{1}{2} \cos 2\sigma \sin 2\tau \right\}$$

To determine p^+ , we use the condition that the momentum is zero.

From (9.52), p^μ is the coefficient of $2\alpha'\tau$ in X^μ .

[So we already have $p^{(2)} = 0$]

Since $X^{(1)} = \frac{X^+ - X^-}{\sqrt{2}}$, $p^{(1)} = 0$ provided $p^+ = \frac{a^2}{2\alpha' p^+}$

i.e. $p^+ = \frac{a}{\sqrt{2\alpha'}}$

This makes $X^+ = \sqrt{2\alpha'} a \tau$

$$X^- = \sqrt{2\alpha'} a \left\{ \tau + \frac{1}{2} \cos 2\sigma \sin 2\tau \right\}$$

which makes $X^0 = \sqrt{2\alpha'} a \left\{ \sqrt{2} \tau + \frac{1}{2\sqrt{2}} \cos 2\sigma \sin 2\tau \right\}$

$$X^1 = \sqrt{2\alpha'} a \left\{ -\frac{1}{2\sqrt{2}} \cos 2\sigma \sin 2\tau \right\}$$

and we already have $X^2 = \sqrt{2\alpha'} a \{ 2 \cos \sigma - \sin \tau \}$

b) For any fixed σ , $\frac{\partial X^0}{\partial \tau} = \sqrt{2\alpha'} a \sqrt{2} \left\{ 1 + \frac{1}{2} \cos 2\sigma \cos 2\tau \right\}$
which is always $> a\sqrt{\alpha'}$ so as τ increases, so does t .

Constant τ corresponds to constant X^0 (independent of σ)

if and only if $\sin 2\tau = 0$ i.e. $\tau = \Gamma \frac{\pi}{2}$, $\Gamma = 0, \pm 1, \pm 2, \dots$

In that case $X^1 = 0$, $X^2 = 2a\sqrt{2\alpha'} \cos \sigma \sin \frac{\Gamma\pi}{2}$

$$X^0 = a\sqrt{2\alpha'} \sqrt{2} \frac{\Gamma\pi}{2}$$

So for $\Gamma = \dots -2, 0, +2, \dots$ strings are

i.e. have zero length and are at $\vec{X} = \vec{0}$.

For $\Gamma = 1, 5, \dots$ and for $\Gamma = 3, 7, \dots$ strings are
along X^2 axis, length $4a\sqrt{2\alpha'}$

$$X^1 = X^2 = 0$$

c) For $\tau \ll 1$, approximate $\sin \tau$ by τ , $\sin 2\tau$ by 2τ and scale all X^μ by $a\sqrt{2\alpha'}$

$$\text{Then } X^0 = t = \sqrt{2} \tau (1 + \frac{1}{2} \cos 2\sigma)$$

$$\text{so } \tau = \frac{\sqrt{2} t}{2 + \cos 2\sigma}$$

$$X^1 = -\frac{\tau}{\sqrt{2}} \cos 2\sigma = -t \frac{\cos 2\sigma}{2 + \cos 2\sigma}$$

$$X^2 = 2\tau \cos \sigma = t \frac{2\sqrt{2} \cos \sigma}{2 + \cos 2\sigma}$$

$$(X^1)^2 + (X^2)^2 = t^2 \frac{\cos^2 2\sigma + 8 \cos^2 \sigma}{(2 + \cos 2\sigma)^2}$$

Since $8 \cos^2 \sigma = 4 \cos 2\sigma + 4$, numerator is $(\cos 2\sigma + 2)^2$

and $(X^1)^2 + (X^2)^2 = t^2$ so string lies on circle radius t ,

centre at $X^1 = X^2 = 0$.

$$\text{When } \sigma = 0 \quad (X^1, X^2) = t \left(-\frac{1}{3}, \frac{2\sqrt{2}}{3} \right)$$

$$\sigma = \frac{\pi}{2} \quad (X^1, X^2) = t (1, 0)$$

$$\sigma = \pi \quad (X^1, X^2) = t \left(-\frac{1}{3}, -\frac{2\sqrt{2}}{3} \right)$$

In fact if we write $(X^1, X^2) = t(\cos \theta, \sin \theta)$

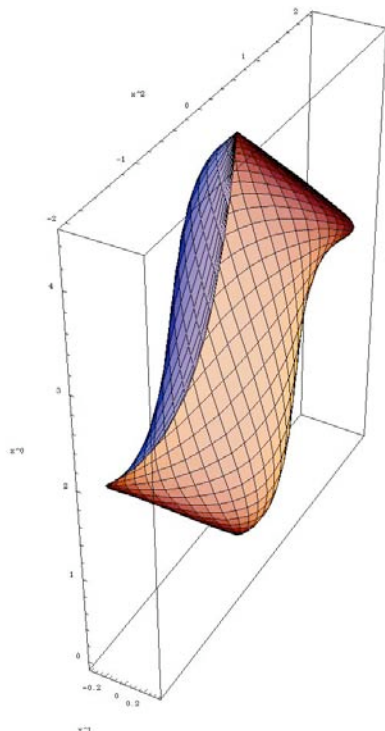
as σ goes from 0 to π , θ goes from γ to 0 to $-\gamma$

where $\cos \gamma = -\frac{1}{3}$ ($\frac{\pi}{2} < \gamma < \pi$)

Since we scaled X^0, X^1, X^2 all by same factor, radius is growing at speed 1 i.e. speed of light

Problem 9.4(d):[‡]

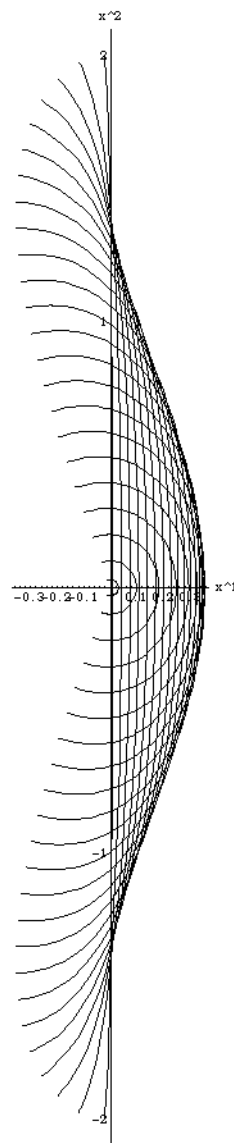
The following 3D graph was constructed by Mathematica:



Since this graph seems hard to interpret, we also constructed a graph which shows the position of the string at fixed values of the time coordinate X^0 . Specifically, the graph on the right shows the string at times

$$\frac{1}{\sqrt{2\alpha'}} \frac{1}{a} X^0 = 0.03, \quad 0.1, \quad 0.2, \quad 0.3, \quad \dots, \quad 2.1, \quad 2.2.$$

Note that the string starts with zero size when $X^0 = 0$, and then grows and becomes straighter. When $\frac{1}{\sqrt{2\alpha'}} \frac{1}{a} X^0 = \pi/\sqrt{2} = 2.221$ it becomes a straight line extending from $\frac{1}{\sqrt{2\alpha'}} \frac{1}{a} X^2 = -2$ to $\frac{1}{\sqrt{2\alpha'}} \frac{1}{a} X^2 = 2$ along the X^2 axis. After that the string returns back to a point, but the new configurations are related to the previous ones by the reflection $X^1 \rightarrow -X^1$.



[§]Solution written by Jeffrey Goldstone.

[‡]Graphics constructed by Barton Zwiebach, with text by Alan Guth.

Problem 9.5: Closed strings in the light-cone gauge[†]

- (a) From Eq. (9.65) of the textbook, using $\beta = 1$ for closed strings, the appropriate linear combinations give

$$\begin{aligned}\dot{X}^- &= \frac{1}{2\alpha'p^+} \left[(\dot{X}^I)^2 + (X^{I'})^2 \right] , \\ X^{-'} &= \frac{1}{\alpha'p^+} \dot{X}^I \cdot X^{I'} .\end{aligned}\tag{1}$$

For this problem the only nonzero function $X^I(\tau, \sigma)$ is

$$X^{(2)}(\tau, \sigma) = \sqrt{2\alpha'} \left[a \sin(\tau - \sigma) + b \cos(\tau - \sigma) + \bar{a} \sin(\tau + \sigma) + \bar{b} \cos(\tau + \sigma) \right] . \tag{2}$$

Since the right-hand sides of Eqs. (1) are quadratic, and $X^2(\tau, \sigma)$ contains four terms, there is cause to look for a simplification. A linear sum of a sine and cosine function of the same argument can be written as a sine function with a phase offset, which converts the 4-term expression to a 2-term expression. To carry out this simplification, introduce new constants R , $\bar{R}$, ϕ , and $\bar{\phi}$, related to the original constants by

$$\begin{aligned}a &= R \cos \phi & b &= R \sin \phi , \\ \bar{a} &= \bar{R} \cos \bar{\phi} & \bar{b} &= \bar{R} \sin \bar{\phi} .\end{aligned}\tag{3}$$

Then, using the trigonometric identity

$$\sin(A + B) = \sin A \cos B + \cos A \sin B , \tag{4}$$

Eq. (2) can be rewritten as

$$X^{(2)}(\tau, \sigma) = \sqrt{2\alpha'} \left[R \sin(\tau - \sigma + \phi) + \bar{R} \sin(\tau + \sigma + \bar{\phi}) \right] . \tag{5}$$

Then

$$\dot{X}^{(2)} = \sqrt{2\alpha'} \left[\bar{R} \cos(\tau + \sigma + \bar{\phi}) + R \cos(\tau - \sigma + \phi) \right] , \tag{6a}$$

$$X^{(2)'} = \sqrt{2\alpha'} \left[\bar{R} \cos(\tau + \sigma + \bar{\phi}) - R \cos(\tau - \sigma + \phi) \right] , \tag{6b}$$

and

$$X^{-'} = \frac{2}{p^+} \left[\bar{R}^2 \cos^2(\tau + \sigma + \bar{\phi}) - R^2 \cos^2(\tau - \sigma + \phi) \right] \tag{7a}$$

$$= \frac{1}{p^+} \left[(\bar{R}^2 - R^2) + \bar{R}^2 \cos 2(\tau + \sigma + \bar{\phi}) - R^2 \cos 2(\tau - \sigma + \phi) \right] , \tag{7b}$$

where we have used the trigonometric identity

$$\cos^2 A = \frac{1 + \cos 2A}{2} . \quad (8)$$

We can now use the fact that $X^-(\tau, \sigma)$ must be periodic in σ with period 2π , so we require

$$X^-(\tau, \sigma + 2\pi) - X^-(\tau, \sigma) = 0 = \int_0^{2\pi} d\sigma X^{-'}(\tau, \sigma) . \quad (9)$$

The cosine terms in Eq. (7b) integrate to zero, so the vanishing of the above integral requires

$$\bar{R}^2 - R^2 = 0 , \quad (10)$$

so from Eq. (3) we have finally

$$\boxed{\bar{a}^2 + \bar{b}^2 = a^2 + b^2} . \quad (11)$$

- (b) We now consider the special case $a = b = \bar{a} = \bar{b} = r$, so $R = \bar{R} = \sqrt{2}r$, and $\phi = \bar{\phi} = \frac{\pi}{4}$. Eq. (7b) becomes

$$X^{-'} = \frac{2r^2}{p^+} \left[\cos 2 \left(\tau + \sigma + \frac{\pi}{4} \right) - \cos 2 \left(\tau - \sigma + \frac{\pi}{4} \right) \right] , \quad (12)$$

which can be integrated to give

$$X^- = \frac{r^2}{p^+} \left[\sin 2 \left(\tau + \sigma + \frac{\pi}{4} \right) + \sin 2 \left(\tau - \sigma + \frac{\pi}{4} \right) \right] + F(\tau) , \quad (13)$$

where $F(\tau)$ is an unknown function. From Eqs. (1), (3), and (6),

$$\dot{X}^- = \frac{4r^2}{p^+} \left[\cos^2 \left(\tau + \sigma + \frac{\pi}{4} \right) + \cos^2 \left(\tau - \sigma + \frac{\pi}{4} \right) \right] \quad (14a)$$

$$= \frac{2r^2}{p^+} \left[2 + \cos 2 \left(\tau + \sigma + \frac{\pi}{4} \right) + \cos 2 \left(\tau - \sigma + \frac{\pi}{4} \right) \right] , \quad (14b)$$

which can be integrated to give

$$X^- = \frac{r^2}{p^+} \left[4\tau + \sin 2 \left(\tau + \sigma + \frac{\pi}{4} \right) + \sin 2 \left(\tau - \sigma + \frac{\pi}{4} \right) \right] + G(\sigma) , \quad (15)$$

where $G(\sigma)$ is an unknown function. Comparing Eqs. (13) and (15), we have finally

$$\boxed{\begin{aligned} X^- &= \frac{r^2}{p^+} \left[4\tau + \sin 2 \left(\tau + \sigma + \frac{\pi}{4} \right) + \sin 2 \left(\tau - \sigma + \frac{\pi}{4} \right) \right] \\ &= \frac{r^2}{p^+} \left[4\tau + \cos 2 (\tau + \sigma) + \cos 2 (\tau - \sigma) \right] . \end{aligned}} \quad (16)$$

To find the mass, we can evaluate the energy-momentum vector and then square it, using $p^2 = -M^2$. Combining Eqs. (11) and (19a), one finds

$$p^\mu = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \dot{X}^\mu . \quad (17)$$

Thus, from Eq. (14b),

$$p^- = \frac{4r^2}{\alpha' p^+} , \quad (18)$$

where the cosine terms integrate to zero over the full period. From Eq. (6a) one sees that

$$p^{(2)} = 0 , \quad (19)$$

since again the cosine terms integrate to zero over the full period. Finally, then,

$$p^2 = -2p^+ p^- + \left(p^{(2)}\right)^2 = -\frac{8r^2}{\alpha'} , \quad (20)$$

and

$$M^2 = \frac{8r^2}{\alpha'} .$$

(21)

[†]Solution written by Alan Guth.

Problem 10.1 Momentum for the classical scalar field

From (10.46) $\vec{P} = - \int d^3x \partial_0 \phi \vec{\nabla} \phi$

with, from (10.33) $\phi = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2E_p}} \left(a(t) e^{i\vec{p} \cdot \vec{x}} + a^*(t) e^{-i\vec{p} \cdot \vec{x}} \right)$

so $\partial_0 \phi = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2E_p}} \left\{ \frac{da}{dt} e^{i\vec{p} \cdot \vec{x}} + \frac{da^*}{dt} e^{-i\vec{p} \cdot \vec{x}} \right\}$

$\vec{\nabla} \phi = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2E_p}} i\vec{p} \left\{ a(t) e^{i\vec{p} \cdot \vec{x}} - a^*(t) e^{-i\vec{p} \cdot \vec{x}} \right\}$

When we take the product and integrate, $\frac{1}{V} \int d^3x e^{\pm 2i\vec{p} \cdot \vec{x}} = 0$
(and $\frac{1}{V} \int d^3x = 1$)

so $\vec{P} = -\frac{i\vec{p}}{2E_p} \left\{ a \frac{da^*}{dt} - a^* \frac{da}{dt} \right\}$

Now we put $a(t) = a_p e^{-iE_p t} + a_{-p}^* e^{iE_p t}$ (not real)
 $i \frac{da}{dt} = E_p \left\{ a_p e^{-iE_p t} - a_{-p}^* e^{iE_p t} \right\}$

so $\vec{P} = \frac{\vec{p}}{2E_p} \left\{ a^* \left(i \frac{da}{dt} \right) + a \left(i \frac{da}{dt} \right)^* \right\}$
 $= \vec{p} \times \text{Real part of } \left\{ a_p^* e^{+iE_p t} + a_{-p} e^{-iE_p t} \right\} \left\{ a_p e^{-iE_p t} - a_{-p}^* e^{iE_p t} \right\}$
 $= \vec{p} \left\{ a_p^* a_p - a_{-p}^* a_{-p} \right\}$

(the other 2 terms give $a_{-p} a_p e^{-2iE_p t} - a_{-p}^* a_p^* e^{2iE_p t}$
which has no real part)

Problem 10.2 Commutator for the quantum scalar field

a) $f(\vec{x}) = \sum_{\vec{p}} f(\vec{p}) e^{i\vec{p} \cdot \vec{x}}$

As in the argument for (10.36)

$$\frac{1}{V} \int d^3x e^{i\vec{p} \cdot \vec{x}} e^{-i\vec{p}' \cdot \vec{x}} = \delta_{\vec{p}, \vec{p}'} = \begin{cases} 0 & \text{if } \vec{p} \neq \vec{p}' \\ 1 & \text{if } \vec{p} = \vec{p}' \end{cases}$$

(where $\vec{p}, \vec{p}'$ both belong to the set of momenta in (10.34))

Hence $\frac{1}{V} \int d^3x f(\vec{x}) e^{-i\vec{p} \cdot \vec{x}} = \sum_{\vec{p}'} f(\vec{p}') \delta_{\vec{p}, \vec{p}'} = f(\vec{p})$

$$\begin{aligned} \text{So } f(\vec{x}) &= \sum_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} \left\{ \frac{1}{V} \int d^3x' f(\vec{x}') e^{-i\vec{p} \cdot \vec{x}'} \right\} \\ &= \int d^3x' f(\vec{x}') \left\{ \sum_{\vec{p}} e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} \right\} \end{aligned}$$

which means $\delta^{(3)}(\vec{x} - \vec{x}') = \frac{1}{V} \sum_{\vec{p}} e^{i\vec{p} \cdot (\vec{x} - \vec{x}')}$

b) From (10.58) $\phi(t, \vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}} e^{-iE_p t + i\vec{p} \cdot \vec{x}} + a_{\vec{p}}^\dagger e^{iE_p t - i\vec{p} \cdot \vec{x}} \right\}$

so $\pi(t, \vec{x}) = \frac{\partial \phi}{\partial t} = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \left(-i\sqrt{\frac{E_p}{2}} \right) \left\{ a_{\vec{p}} e^{-iE_p t + i\vec{p} \cdot \vec{x}} - a_{\vec{p}}^\dagger e^{iE_p t - i\vec{p} \cdot \vec{x}} \right\}$

When we calculate $[\phi, \pi]$ only the terms with $[a_{\vec{p}}, a_{\vec{p}}^\dagger] = 1$

and $[a_{\vec{p}}^\dagger, a_{\vec{p}}] = -1$ are non-zero, so

$$\begin{aligned} [\phi(\vec{x}, t), \pi(\vec{x}', t)] &= \frac{1}{V} \sum_{\vec{p}} \left(\frac{-i}{2} \right) \left\{ -e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} - e^{-i\vec{p} \cdot (\vec{x} - \vec{x}')} \right\} \\ &= i \delta^{(3)}(\vec{x} - \vec{x}') \end{aligned}$$

(Note that the dependence on t disappeared. In general $[\phi(\vec{x}, t), \pi(\vec{x}', t')]$ is a function of $\vec{x} - \vec{x}', t - t'$)

Problem 10.3. *Light-cone components of Lorentz tensors.*

(a) Since every component of A equals the corresponding component of B , we have that

$$A^\pm = \frac{1}{\sqrt{2}}(A^0 \pm A^1) = \frac{1}{\sqrt{2}}(B^0 \pm B^1) = B^\pm$$

The fact that $A^I = B^I$ follows from $A^\mu = B^\mu$ setting $\mu = I$.

(b) The fact that our definition works for tensors of the form $R^{\mu\nu} = A^\mu B^\nu$ implies that

$$\begin{aligned} R^{++} &= A^+ B^+ = \frac{1}{\sqrt{2}}(A^0 + A^1) \frac{1}{\sqrt{2}}(B^0 + B^1) \\ &= \frac{1}{2}(A^0 B^0 + A^0 B^1 + A^1 B^0 + A^1 B^1) \\ &= \frac{1}{2}(R^{00} + R^{01} + R^{10} + R^{11}) \end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned} R^{+-} &= A^+ B^- = \frac{1}{\sqrt{2}}(A^0 + A^1) \frac{1}{\sqrt{2}}(B^0 - B^1) \\ &= \frac{1}{2}(A^0 B^0 - A^0 B^1 + A^1 B^0 - A^1 B^1) \\ &= \frac{1}{2}(R^{00} - R^{01} + R^{10} - R^{11}) \end{aligned} \tag{2}$$

$$\begin{aligned} R^{-+} &= A^- B^+ = \frac{1}{\sqrt{2}}(A^0 - A^1) \frac{1}{\sqrt{2}}(B^0 + B^1) \\ &= \frac{1}{2}(A^0 B^0 + A^0 B^1 - A^1 B^0 - A^1 B^1) \\ &= \frac{1}{2}(R^{00} + R^{01} - R^{10} - R^{11}) \end{aligned} \tag{3}$$

$$\begin{aligned} R^{--} &= A^- B^- = \frac{1}{\sqrt{2}}(A^0 - A^1) \frac{1}{\sqrt{2}}(B^0 - B^1) \\ &= \frac{1}{2}(A^0 B^0 - A^0 B^1 - A^1 B^0 + A^1 B^1) \\ &= \frac{1}{2}(R^{00} - R^{01} - R^{10} + R^{11}) \end{aligned} \tag{4}$$

In general we may write the change of coordinates of a vector field as $A^{\hat{\mu}} = M_{\nu}^{\hat{\mu}} A^{\nu}$ where the hatted index is a light-cone index and $M_0^{\pm} = \frac{1}{\sqrt{2}}$ and $M_1^{\pm} = \pm \frac{1}{\sqrt{2}}$ and $M_J^{\hat{I}} = \delta_J^{\hat{I}}$. Since each index of a tensor transforms like a vector, we can write

$$R^{\hat{\mu}\hat{\nu}} = M_{\mu}^{\hat{\mu}} M_{\nu}^{\hat{\nu}} R^{\mu\nu}$$

This immediately implies that if $R^{\mu\nu} = S^{\mu\nu}$ that

$$R^{\hat{\mu}\hat{\nu}} = M_{\mu}^{\hat{\mu}} M_{\nu}^{\hat{\nu}} R^{\mu\nu} = M_{\mu}^{\hat{\mu}} M_{\nu}^{\hat{\nu}} S^{\mu\nu} = S^{\hat{\mu}\hat{\nu}}$$

(c) We have

$$\begin{aligned}\eta^{++} &= \frac{1}{2}(\eta^{00} + \eta^{01} + \eta^{10} + \eta^{11}) = \frac{1}{2}(-1 + 0 + 0 + 1) = 0 \\ \eta^{+-} &= \frac{1}{2}(\eta^{00} - \eta^{01} + \eta^{10} - \eta^{11}) = \frac{1}{2}(-1 - 0 + 0 - 1) = -1 \\ \eta^{-+} &= \frac{1}{2}(\eta^{00} + \eta^{01} - \eta^{10} - \eta^{11}) = \frac{1}{2}(-1 + 0 - 0 - 1) = -1 \\ \eta^{--} &= \frac{1}{2}(\eta^{00} - \eta^{01} - \eta^{10} + \eta^{11}) = \frac{1}{2}(-1 - 0 - 0 + 1) = 0\end{aligned}$$

(d) We find

$$F^{+-} = \frac{1}{2}(F^{00} - F^{01} + F^{10} - F^{11}) = F^{10} = -E^x.$$

Similarly

$$F^{+I} = \frac{1}{\sqrt{2}}(F^{0I} + F^{1I}) = \frac{1}{\sqrt{2}}(E^I + \epsilon^{1IJ} B_J)$$

and

$$F^{-I} = \frac{1}{\sqrt{2}}(F^{0I} - F^{1I}) = \frac{1}{\sqrt{2}}(E^I - \epsilon^{1IJ} B_J)$$

Finally, there is only one non-zero component of F^{IJ} given by $F^{23} = -F^{32} = B_x$.

Problem 10.4 Constant electric field in light-cone gauge

$$E_x = F^{01} = \partial^0 A^1 - \partial^1 A^0$$

Gauge $A^+ = 0$ means $A^1 = -A^0$ and $A^- = \frac{1}{\sqrt{2}}(A^0 - A^1) = \sqrt{2} A^0$

so $E_x = -\frac{1}{\sqrt{2}}(\partial^0 + \partial^1)A^- = -\partial^+ A^- = +\frac{\partial A^-}{\partial x^-}$

(remember $\eta^{+-} = \eta^{-+} = -1$)

[Alternative method, using light-cone components only.]

Since $x^0 = \frac{x^+ + x^-}{\sqrt{2}}$ and $x^1 = \frac{x^+ - x^-}{\sqrt{2}}$, for 2 vectors x^μ, y^μ

we have $x^0 y^1 = \frac{1}{2}(x^+ y^+ - x^+ y^- + x^- y^+ - x^- y^-)$

Hence for any tensor $T^{01} = \frac{1}{2}(T^{++} - T^{+-} + T^{-+} - T^{--})$

and for an antisymmetric tensor, $F^{01} = -F^{+-}$

So $F^{+-} = -E_x = \partial^+ A^- - \partial^- A^+$. With $A^+ = 0$, and $\partial^+ = -\frac{\partial}{\partial x^-}$

this gives $\frac{\partial A^-}{\partial x^-} = E_x$, as above]

Take $A^- = E_x x^-$, $A^+ = A^I = 0$

Since $\frac{\partial A^-}{\partial x^-}$ is the only non-zero derivative, this gives $\vec{E} = (E_x, 0, \dots)$ and zero magnetic field.

We can still add a pure gauge, transformation $A^\mu = \partial^\mu \lambda$

We want $A^+ = 0$ so $\frac{\partial \lambda}{\partial x^-} = 0$, $\lambda = \lambda(x^+, x^I)$

This makes $A^+ = 0$, $A^- = E_x x^- + \frac{\partial \lambda}{\partial x^+}$, $A^I = \frac{\partial \lambda}{\partial x^I}$

[We cannot use $p^+ \neq 0$ here: $A(p)$ contains derivatives of $\delta^{(D)}(p)$!]

Problem 10.5 Gravitational fields that are pure gauge

A pure gauge $h_{\mu\nu}$ is one that is equivalent to zero by a gauge transformation i.e. in terms of $\vec{p}$ components, from (10.90),

$$h^{\mu\nu}(\vec{p}) = i\vec{p}^\mu \vec{E}^\nu + i\vec{p}^\nu \vec{E}^\mu(\vec{p})$$

Equations of motion (10.89), $\nabla \cdot \vec{p} \neq 0$, give

$$h^{\mu\nu}(\vec{p}) = \frac{1}{p^2} \left\{ p_\alpha p^\mu h^{\nu\alpha} + p_\alpha p^\nu h^{\mu\alpha} - p^\mu p^\nu h^\alpha_\alpha \right\}$$

$$\text{Define } i\vec{E}^\mu(\vec{p}) = \frac{1}{p^2} \left\{ p_\alpha h^{\mu\alpha} - \frac{1}{2} p^\mu h^\alpha_\alpha \right\}$$

and we see at once that $h^{\mu\nu}$ is pure gauge

Problem 10.6 Kalb-Ramond field

$$B_{\mu\nu} = -B_{\nu\mu}, \quad H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$

$$a) \quad H_{\nu\mu\rho} = \partial_\nu B_{\mu\rho} + \partial_\mu B_{\rho\nu} + \partial_\rho B_{\nu\mu}$$

$$= -\partial_\nu B_{\rho\mu} - \partial_\mu B_{\rho\nu} - \partial_\rho B_{\nu\mu} = -H_{\mu\nu\rho}$$

so H is antisymmetric in the first 2 indices, and from its definition, is unchanged by a cyclic permutation of the indices, i.e. $H_{\mu\nu\rho} = H_{\nu\rho\mu} = H_{\rho\mu\nu}$. We can use these to move any pair of indices to the front and so H must be antisymmetric under interchange of any pair.

(If you don't like these words, just write out the formulas)

$$\text{If } \delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu, \quad \delta(\partial_\mu B_{\nu\rho}) = \partial_\mu \partial_\nu \epsilon_\rho - \partial_\mu \partial_\rho \epsilon_\nu$$

$$\delta H_{\mu\nu\rho} = \partial_\mu \partial_\nu \epsilon_\rho - \partial_\mu \partial_\rho \epsilon_\nu + \partial_\nu \partial_\rho \epsilon_\mu - \partial_\nu \partial_\mu \epsilon_\rho + \partial_\rho \partial_\mu \epsilon_\nu - \partial_\rho \partial_\nu \epsilon_\mu$$

$$= 0$$

$$b) \quad \text{If } \epsilon'_\mu = \epsilon_\mu + \partial_\mu \lambda, \quad \delta' B_{\mu\nu} = \delta B_{\mu\nu} + \partial_\mu \partial_\nu \lambda - \partial_\nu \partial_\mu \lambda$$

$$= \delta B_{\mu\nu}$$

c) In momentum space (b) becomes $E'^\mu(p) = E^\mu(p) + i p^\mu \lambda(p)$

(i is put in to make reality properties come out right)

$$E^{+\prime}(p) = E^+(p) + i p^+ \lambda(p) \quad \text{so we can make } E^{+\prime} = 0$$

by choosing $\lambda = \frac{i}{p^+} E^+(p)$ (and as for the other cases, we assume $p^+ \neq 0$)

$$d) S = -\frac{1}{6} \int d^D x H^{\mu\nu\rho} H^{\mu\nu\rho}$$

$$\delta S = -\frac{1}{3} \int d^D x H^{\mu\nu\rho} (\partial_\mu \delta B_{\nu\rho} + \partial_\nu \delta B_{\rho\mu} + \partial_\rho \delta B_{\mu\nu})$$

$$= - \int d^D x H^{\mu\nu\rho} \partial_\rho \delta B_{\mu\nu} \text{ (cyclic symmetry)} = \int d^D x \delta B_{\mu\nu} \partial_\rho H^{\mu\nu\rho}$$

This is to be zero for all antisymmetric $\delta B_{\mu\nu}$, so since $\partial_\rho H^{\mu\nu\rho}$ is antisymmetric,

$$\partial_\rho H^{\mu\nu\rho} = 0$$

$$\text{i.e. } \partial^\rho \partial_\mu B_{\nu\rho} + \partial^\rho \partial_\nu B_{\rho\mu} + \partial^\rho \partial_\rho B_{\mu\nu} = 0$$

$$\text{i.e. } \partial^\rho \partial_\rho B_{\mu\nu} + \partial_\mu (\partial^\rho B_{\nu\rho}) - \partial_\nu (\partial^\rho B_{\rho\mu}) = 0$$

$$\text{In p-space this becomes } p^2 B^{\mu\nu}(p) = p^\nu p_\alpha B^{\mu\alpha}(p) - p^\mu p_\alpha B^{\nu\alpha}(p)$$

e) Try to choose $\epsilon^\mu(p)$ to make $B^{+\mu}(p) = 0$, i.e. $B^{+-} = B^{+I} = 0$ ($B^{++} = 0$ by antisymmetry). From (c), we can take $\epsilon^+(p) = 0$

Then $\delta B^{+\mu} = i p^\mu \epsilon^+$, $\mu = -, I$ so we can make $B^{+\mu} = 0$ (assuming $p^+ \neq 0$) and this fixes the gauge completely.

In this gauge, define $p_\alpha B^{\mu\alpha} = A^\mu$ so that $A^+ = 0$

$$A^- = p_- B^{-I}, \quad A^I = p_- B^{I-} + p_J B^{IJ} = p^+ B^{-I} + p_J B^{IJ}$$

$$\text{Equations of motion are } p^2 B^{\mu\nu} = p^\nu A^\mu - p^\mu A^\nu$$

$$\mu\nu = +- \text{ gives } p^+ A^- = 0 \text{ so } A^- = 0$$

$$\mu\nu = +I \text{ gives } p^+ A^I = 0 \text{ so } A^I = 0, \text{ so } A^\mu = 0$$

$$\text{and so } p^2 B^{\mu\nu} = 0$$

$$A^I = 0 \text{ gives } B^{-I} = -\frac{1}{p^+} p_J B^{IJ}, \text{ and then } A^- = -\frac{1}{p^+} p_- p_J B^{IJ} = 0$$

$$\text{Independent degrees of freedom are } B^{IJ} = -B^{JI}, \text{ with } p^2 B^{IJ} = 0$$

(in D spacetime dimensions, there are $\frac{1}{2}(D-2)(D-3)$ d.f.)

f) Exactly as in the argument leading to (10.88), one-particle states come from expanding the independent fields B^{IJ} in plane waves,

$$\text{giving } \sum_{IJ} \hat{S}_{IJ} a_p^{IJ\dagger} |\Omega\rangle \text{ with } a^{IJ\dagger} = -a^{JI\dagger} \text{ so } \hat{S}_{IJ} = -\hat{S}_{JI}.$$

Problem 10.7. Massive vector field.

We have been given the action

$$S = \int d^D x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + m A^\mu \partial_\mu \phi \right). \quad (1)$$

We note that the last three terms in the action combine into a perfect square:

$$S = \int d^D x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu \phi - m A_\mu)^2 \right). \quad (2)$$

(a) We consider the candidate gauge transformations

$$\delta A_\mu = \partial_\mu \epsilon, \quad \delta \phi = b m \epsilon.$$

The variation of the action is computed using (2), in which the first term inside the parentheses is gauge invariant. It suffices if $\partial_\mu \phi - m A_\mu$ is gauge invariant, thus

$$\delta(\partial_\mu \phi - m A_\mu) = b m \partial_\mu \epsilon - m \partial_\mu \epsilon = 0,$$

which fixes $b = 1$. So the gauge transformations are

$$\delta A_\mu = \partial_\mu \epsilon, \quad \delta \phi = m \epsilon. \quad (3)$$

(b) The variation of the action (1) gives:

$$\delta S = \int d^D x \left(\partial_\nu (\delta A_\mu) F^{\mu\nu} - m^2 (\delta A_\mu) A^\mu - \partial_\mu (\delta \phi) \partial^\mu \phi + m \delta A^\mu \partial_\mu \phi + m A^\mu \partial_\mu \delta \phi \right).$$

We integrate by parts all derivatives that act on variations:

$$\delta S = \int d^D x \left(-\delta A_\mu \partial_\nu F^{\mu\nu} - \delta A_\mu m^2 A^\mu + \delta A_\mu m \partial^\mu \phi + \delta \phi \partial^2 \phi - \delta \phi m \partial \cdot A \right).$$

The equations of motion are therefore

$$-\partial_\nu F^{\mu\nu} - m^2 A^\mu + m \partial^\mu \phi = 0, \quad \partial^2 \phi - m \partial \cdot A = 0. \quad (4)$$

One readily verifies that the equations of motion are gauge invariant. This is good, but not strictly necessary; it suffices for consistency if the variations vanish when the equations of motion hold.

(c) The gauge transformation $\delta \phi = m \epsilon$ allows us to gauge away ϕ : setting $\epsilon = -\phi/m$, one finds $\phi \rightarrow \phi + \epsilon m = 0$. With $\phi = 0$ the equations of motion (4) become:

$$-\partial_\nu F^{\mu\nu} - m^2 A^\mu = 0, \quad \partial \cdot A = 0.$$

More explicitly:

$$-\partial^\mu(\partial \cdot A) + \partial^2 A^\mu - m^2 A^\mu = 0, \quad \partial \cdot A = 0.$$

This is equivalent to

$$\partial^2 A^\mu - m^2 A^\mu = 0, \quad \partial \cdot A = 0. \quad (5)$$

The above are the equations of motion in the gauge where the scalar field has disappeared.

(d) In momentum space equations (5) become

$$(p^2 + m^2)A^\mu = 0, \quad p \cdot A = 0. \quad (6)$$

When $p^2 \neq -m^2$ the first equation gives $A^\mu(p) = 0$. Consider now $p^2 = -m^2$, in which case the first equation is satisfied for arbitrary $A^\mu(p)$. Using a suitable Lorentz transformation any timelike vector p^μ can be rotated into a vector with only a time component. We can thus assume that $p^\mu = (m, \vec{0})$ since this ansatz satisfies $p^2 = -m^2$. The second equation gives $p \cdot A = p_0 A^0 + \vec{p} \cdot \vec{A} = -m A^0 = 0$. So we conclude that $A^0 = 0$. The remaining $D - 1$ components A^i with $i = 1, \dots, D - 1$ remain unconstrained. We have shown that the massive vector field has $D - 1$ degrees of freedom.

Problem 11.1 Equation of motion for Heisenberg operators

$$\xi(t) = e^{iHt} \xi e^{-iHt} \quad \text{with } H = H(p, q)$$

$$i \frac{d\xi(t)}{dt} = e^{iHt} \xi H e^{-iHt} - e^{iHt} H \xi e^{-iHt}$$

$$\left[\text{using } \frac{d}{dt} e^{iHt} = iH e^{iHt} = i e^{iHt} H \right]$$

Now we use the fact that $e^{iHt} A B \dots Z e^{-iHt}$

$$= (e^{iHt} A e^{-iHt}) (e^{iHt} B e^{-iHt}) \dots (e^{iHt} Z e^{-iHt})$$

so that whatever $H(p, q)$ is, $e^{iHt} H(p, q) e^{-iHt} =$

$$H(e^{iHt} p e^{-iHt}, e^{iHt} q e^{-iHt})$$

so that

$$i \frac{d\xi(t)}{dt} = (e^{iHt} \xi e^{-iHt}) H(e^{iHt} p e^{-iHt}, e^{iHt} q e^{-iHt}) \leftrightarrow$$

$$= [\xi(t), H(p(t), q(t))].$$

Problem 11.2 Heisenberg operators and time-dependent Hamiltonians

a) $|\psi, t\rangle = U(t)|\psi\rangle$ should satisfy the Schrödinger equation
 $i\frac{d}{dt}|\psi, t\rangle = H(p, q; t)|\psi, t\rangle$ for any $|\psi\rangle$, which can only be
 if $i\frac{d}{dt}U(t) = H(p, q; t)U(t)$ (and with $|\psi, 0\rangle = |\psi\rangle$, $U(0) = 1$)

b) $\xi(t) = U^{-1}(t)\xi U(t)$
 $i\frac{d\xi(t)}{dt} = U^{-1}(t)\xi\left(i\frac{dU}{dt}\right) + \left(i\frac{dU^{-1}}{dt}\right)\xi U$
 Since $U^{-1}U = 1$, $\frac{dU^{-1}}{dt}U + U^{-1}\frac{dU}{dt} = 0$ so $i\frac{dU^{-1}}{dt} = -iU^{-1}\frac{dU}{dt}U^{-1}$
 $= -U^{-1}HUU^{-1} = -U^{-1}(t)H(p, q; t)$

So $i\frac{d\xi(t)}{dt} = U^{-1}\xi H(p, q; t)U - U^{-1}H(p, q; t)\xi U$

Once again as in problem 6, $U^{-1}AB\cdots ZU = (U^{-1}AU)(U^{-1}BU)\cdots(U^{-1}ZU)$

so $i\frac{d\xi(t)}{dt} = (U^{-1}\xi U)H(U^{-1}pU, U^{-1}qU; t) - \Leftrightarrow$
 $= [\xi(t), H(p(t), q(t); t)]$.

c) $[\alpha_1(t), \alpha_2(t)] = (U^{-1}\alpha_1U)(U^{-1}\alpha_2U) - (U^{-1}\alpha_2U)(U^{-1}\alpha_1U)$
 $= U^{-1}(\alpha_1\alpha_2 - \alpha_2\alpha_1)U = U^{-1}\alpha_3U = \alpha_3(t)$.

11.3 Classical dynamics in Hamiltonian language.

As stated, to derive the classical equations of motion in Hamiltonian language we vary

$$\delta \int_{t_i}^{t_f} dt \left(p\dot{q} - H(p, q; t) \right) = \int_{t_i}^{t_f} dt \left(\delta p \frac{dq}{dt} + p \frac{d\delta q}{dt} - \frac{\partial H}{\partial p} \delta p - \frac{\partial H}{\partial q} \delta q \right),$$

with independent variations $\delta q(t)$ and $\delta p(t)$ that vanish at t_i and t_f . Replacing

$$p \frac{d\delta q}{dt} = \frac{d(\delta q p)}{dt} - \delta q \frac{dp}{dt},$$

and integrating the total derivative, the variation becomes

$$\left[\delta q(t) p(t) \right]_{t_i}^{t_f} + \int_{t_i}^{t_f} dt \left(\delta p \left[\frac{dq}{dt} - \frac{\partial H}{\partial p} \right] - \delta q \left[\frac{dp}{dt} + \frac{\partial H}{\partial q} \right] \right).$$

The terms at initial and final times vanish because of the boundary conditions on the variations, and the condition that the variations vanish for otherwise arbitrary $\delta q(t)$ and $\delta p(t)$ give

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

Using these equations we readily find that

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial q} \frac{dq}{dt} + \frac{\partial v}{\partial p} \frac{dp}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial v}{\partial p} \frac{\partial H}{\partial q} = \frac{\partial v}{\partial t} + \{v, H\}.$$

This is what we wanted to show. The Hamiltonian generates time evolution in classical mechanics via the Poisson brackets.

Problem 11.4 Reparametrization symmetries of the point particle

$$L = -m \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \quad \delta x^\mu = \lambda \dot{x}^\mu$$

Since L depends only on $\dot{x}$, $\delta L = \frac{\partial L}{\partial \dot{x}^\mu} \frac{d}{d\tau} (\lambda \dot{x}^\mu)$

$$= \lambda \frac{\partial L}{\partial \dot{x}^\mu} \ddot{x}^\mu + \dot{\lambda} \dot{x}^\mu \frac{\partial L}{\partial \dot{x}^\mu} = \lambda \dot{L} + \dot{\lambda} \dot{x}^\mu \frac{\partial L}{\partial \dot{x}^\mu}$$

Since L is homogeneous of degree 1 in $\dot{x}^\mu$ (ie $L(k\dot{x}) = kL(\dot{x})$)

$$\dot{x}^\mu \frac{\partial L}{\partial \dot{x}^\mu} = L \quad (\text{or check this explicitly}) \quad \text{and so} \quad \delta L = \frac{d}{d\tau} (\lambda L)$$

Problem 8.5 tells us that $Q_\lambda = \frac{\partial L}{\partial \dot{x}^\mu} \lambda \dot{x}^\mu - \lambda L$

is the associated conserved charge and we see it is identically zero.
We also see that $Q_\lambda = \lambda H$ with the canonical definition of H from L
and that $H \equiv 0$.

Problem 11.5 Lorentz generators and Lorentz algebra

Here we use the Lorentz covariant quantization with basic commutation relations (11.58), (11.59) $[x^\mu, p^\nu] = i\eta^{\mu\nu}$

$$[x^\mu, x^\nu] = [p^\mu, p^\nu] = 0 \quad (\text{all operators at same } \tau)$$

and $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$

a) $[M^{\mu\nu}, p^\rho] = [x^\mu, p^\rho] p^\nu - [x^\nu, p^\rho] p^\mu$
 $= i(\eta^{\mu\rho} p^\nu - \eta^{\nu\rho} p^\mu) \quad (\text{cf. 11.78})$

b) $[M^{\mu\nu}, M^{\rho\sigma}] = [M^{\mu\nu}, x^\rho p^\sigma - x^\sigma p^\rho]$
 $= [M^{\mu\nu}, x^\rho] p^\sigma + x^\rho [M^{\mu\nu}, p^\sigma] - [M^{\mu\nu}, x^\sigma] p^\rho - x^\sigma [M^{\mu\nu}, p^\rho]$
 $= i\{ \eta^{\mu\rho} x^\nu p^\sigma - \eta^{\nu\rho} x^\mu p^\sigma + \eta^{\mu\sigma} x^\rho p^\nu - \eta^{\nu\sigma} x^\rho p^\mu$
 $\quad - \eta^{\mu\sigma} x^\nu p^\rho + \eta^{\nu\sigma} x^\mu p^\rho - \eta^{\mu\rho} x^\sigma p^\nu + \eta^{\nu\rho} x^\sigma p^\mu \}$
 $= i\{ \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\mu\sigma} M^{\rho\nu} - \eta^{\nu\sigma} M^{\rho\mu} \} \quad (11.81)$
 $= i\{ \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\sigma} M^{\mu\rho} \}$ (which makes the antisymmetry in $\mu, \nu; \rho, \sigma$ manifest)

c) With μ, ν, ρ, σ light cone indices, we use (11.81) with $\eta^{+-} = -1, \eta^{++} = \eta^{--} = 0, \eta^{+I} = \eta^{-I} = 0, \eta^{IJ} = \delta^{IJ}$

[In the following equations with $\pm$, read either all the top row or all the bottom row]

$$[M^{\pm I}, M^{JK}] = i \left\{ -\delta^{IJ} M^{\pm K} + \delta^{IK} M^{\pm J} \right\}$$

$$[M^{\pm I}, M^{\mp J}] = i \left\{ -M^{IJ} + \delta^{IJ} M^{\pm \mp} \right\} = i \left\{ M^{IJ} \pm \delta^{IJ} M^{\mp \mp} \right\}$$

$$[M^{+-}, M^{+I}] = i \left\{ +M^{+I} \right\}$$

$$[M^{+-}, M^{-I}] = i \left\{ -M^{-I} \right\}$$

$$\text{so } [M^{+-}, M^{\pm I}] = \pm i M^{\pm I}$$

$$[M^{\pm I}, M^{\pm J}] = i \left\{ \delta^{IJ} M^{\pm \pm} \right\} = 0.$$

Problem 11.6 Light-cone gauge commutator $[M^{-I}, M^{-J}]$ for the particle

Here we use light-cone gauge operators, in contrast to problem 11.5 where we used invariant operators (which we have not constructed)

a) From (11.88) $M^{-I} = x_0^- p^I - \frac{1}{2}(x_0^I \bar{p} + \bar{p} x_0^I)$

$$= x_0^- p^I - x_0^I \bar{p} + \frac{1}{2} [x_0^I, \bar{p}]$$

Since $\bar{p} = \frac{1}{2p^+}(p^I p^I + m^2)$ and $[x_0^I, p^J] = i \delta^{IJ}$

this gives $M^{-I} = (x_0^- p^I - x_0^I \bar{p}) + \frac{1}{2} i \frac{p^I}{p^+}$

$$[M^{-I}, M^{-J}] = [x_0^- p^I - x_0^I \bar{p}, x_0^- p^J - x_0^J \bar{p}]$$

$$+ \frac{1}{2} i [x_0^- p^I - x_0^I \bar{p}, \frac{p^J}{p^+}] - \frac{1}{2} i [x_0^- p^J - x_0^J \bar{p}, \frac{p^I}{p^+}] \dots (1)$$

(Since p^I, p^J, p^+ commute so the commutator of the 2 last terms gives zero)

The second term is $\frac{1}{2} i [x_0^I, \frac{1}{p^+}] p^I p^J - \frac{1}{2} i [x_0^I, p^J] \frac{p^I}{p^+}$

($p^I, p^J, p^+, \bar{p}$ all commute), $[x_0^I, p^J] = i \delta^{IJ}$ so this is symmetric in I, J and so the second and third terms in (1) cancel.

b) The first term in (1) has non-zero contributions from $[x_0^-, \bar{p}]$, $[x_0^I, p^J]$, $[x_0^I, \bar{p}]$. We get

$$\begin{aligned}
 [M^{-I}, M^{-J}] &= x^I [\bar{x}, \bar{p}] p^J - x^J [\bar{x}, \bar{p}] p^I \\
 &\quad - x^I [\bar{x}^J, \bar{p}] p^- + x^J [\bar{x}^I, \bar{p}] p^- \\
 &\quad + x^- [\bar{x}^J, p^I] p^- - x^- [\bar{x}^I, p^J] p^-
 \end{aligned}$$

(all x should be x_0)

$$\text{From (11.69)} \quad [\bar{x}, \bar{p}] = i \frac{p^-}{p^+}$$

$$\text{and also } [x^I, p^-] = [x^I, \frac{p^J p^J + m^2}{2p^+}] = i \frac{p^I}{p^+}.$$

$$\text{So } [M^{-I}, M^{-J}] = i \left\{ x^I \frac{p^- p^J}{p^+} - x^J \frac{p^- p^I}{p^+} + \delta^{IJ} x^- p^- \right.$$

$$\left. - I \leftrightarrow J \right\}$$

$$= 0.$$

Problem 11.4 Transformations generated by M^{+-} , M^{-I}

$$a) M^{+-} = -\frac{1}{2} (x_0^- p^+ + p^+ x_0^-) = -x_0^- p^+ - \frac{1}{2} i$$

(and we can drop the $-\frac{1}{2}i$ in calculating $[M^{+-}, \cdot]$)

$$x^+(\tau) = \frac{p^+ \tau}{m^2} \quad (11.29) \quad \text{and} \quad [x_0^-, p^+] = -i \quad (11.27)$$

$$\text{so } [M^{+-}, x^+(\tau)] = +i x^+(\tau)$$

$$x^-(\tau) = x_0^- + \frac{p^- \tau}{m^2}$$

$$\begin{aligned} [M^{+-}, x^-(\tau)] &= x_0^- [x_0^-, p^+] - \frac{\tau}{m^2} [x_0^-, p^-] p^+ \\ &= -i x_0^- - \frac{\tau}{m^2} \frac{i p^-}{p^+} p^+ \\ &= -i x^-(\tau) \end{aligned}$$

$$[M^{+-}, x^I(\tau)] = [-x_0^- p^+, x_0^I + p^I \frac{\tau}{m^2}] = 0$$

M^{+-} generates transformations by $\delta x^\mu = [-i \epsilon M^{+-}, x^\mu]$

$$\text{so } \delta x^+ = \epsilon x^+, \quad \delta x^- = -\epsilon x^-, \quad \delta x^I = 0$$

There are the expected Lorentz transformations. [Compare with

$$(11.75) \text{ with } E_{+-} = -E_{-+} = \epsilon \quad \text{and so } \epsilon^{+-} = E_{-+} = -\epsilon_{+-} = -\epsilon$$

$$\delta x^+ = \epsilon^{+-} x_- = -\epsilon x_- = \epsilon x^+$$

$$\delta x^- = \epsilon^{-+} x_+ = \epsilon x_+ = -\epsilon x^-$$

$$[\epsilon M^{+-} \text{ is a boost in the } 1 \text{ direction: } \delta x^0 = \epsilon x^1, \quad \delta x^1 = \epsilon x^0]$$

b) As in 11.6a), $M^{-I} = x_0^- p^I - x_0^I p^- + \frac{1}{2} i \frac{p^I}{p^+}$

With $x^+(\tau) = \frac{p^+ \tau}{m^2}$, $[M^{-I}, x^+] = \frac{\tau}{m^2} [x_0^-, p^+] p^I$
 $= -i \frac{\tau}{m^2} p^I = -i (x^I - x_0^I)$

With $x^-(\tau) = x_0^- + \frac{p^- \tau}{m^2}$, $[M^{-I}, x^-] = -x_0^I [p^-, x_0^-]$
 $+ \frac{1}{2} i [\frac{1}{p^+}, x_0^-] p^I + \frac{\tau}{m^2} [x_0^-, p^-] p^I - \frac{\tau}{m^2} [x_0^I, p^-] p^-$
 $= i x_0^I \frac{p^-}{p^+} - \frac{1}{2} i i \frac{1}{p^+} p^I + \frac{i \tau}{m^2} \frac{p^-}{p^+} p^I - \frac{i \tau}{m^2} \frac{p^I}{p^+} p^-$
 $= \frac{1}{2} i \frac{1}{p^+} (x_0^I p^- + p^- x_0^I) \quad (\text{using 11.68 and 11.69})$

With $x^J(\tau) = x_0^J + p^J \frac{\tau}{m^2}$, $[M^{-I}, x^J] = x_0^- [p^I, x_0^J]$
 $- \frac{\tau}{m^2} [x_0^I, p^J] p^- - x_0^I [p^-, x_0^J] + \frac{1}{2} i [p^I, x_0^J] \frac{1}{p^+}$
 $= -i \delta^{IJ} (x_0^- + \frac{\tau}{m^2} p^-) + i x_0^I \frac{p^J}{p^+} - \frac{1}{2} i i \frac{\delta^{IJ}}{p^+}$
 $= -i \delta^{IJ} x^-(\tau) + \frac{i}{2 p^+} (x_0^I p^J + p^J x_0^I)$

The expected Lorentz transformations are

$$[M^{-I}, x^+] = i \eta^{-+} x^I = -i x^I$$

$$[M^{-I}, x^-] = 0$$

$$[M^{-I}, x^J] = -i \eta^{IJ} x^- = -i \delta^{IJ} x^-$$

We see that the extra terms are

$$[M^I, x^+]_{\text{extra}} = i x_0^I$$

$$[M^{-I}, x^-]_{\text{extra}} = \frac{i}{2p^+} (x_0^I p^- + p^- x_0^I)$$

$$[M^{-I}, x^J]_{\text{extra}} = \frac{i}{2p^+} (x_0^I p^J + p^J x_0^I)$$

Compare these to $\frac{dx^+}{d\tau} = \frac{p^+}{m^2}$, $\frac{dx^-}{d\tau} = \frac{p^-}{m^2}$, $\frac{dx^J}{d\tau} = \frac{p^J}{m^2}$

and we see that the extra terms are $\frac{im^2}{2} \left(\frac{x_0^I}{p^+} \frac{dx^\mu}{d\tau} + \frac{dx^\mu}{d\tau} \frac{x_0^I}{p^+} \right)$

($\mu = +, -, J$) corresponding to a reparameterization by $\lambda = \frac{m^2 x_0^I}{p^+}$

(since x_0^I does not commute with $\frac{dx^\mu}{d\tau}$ we are not surprised by the 'hermiticized' version)

$$c) [M^{-I}, p^+] = [x_0^-, p^+] p^I = -i p^I$$

and we found $[M^{-I}, x^+(\tau)] = -\frac{i\tau}{m^2} p^I$

Hence $[M^{-I}, x^+(\tau) - \frac{\tau}{m^2} p^+] = 0$

The light-cone gauge condition is $x^+(\tau) = \frac{p^+ \tau}{m^2}$

and so is preserved by transformations generated by M^{-I} .

Problem 12.1 Heisenberg equation for momentum density

$$i \frac{d p^{\tau I}(\tau, \sigma)}{d\tau} = [p^{\tau I}(\tau, \sigma), H(\tau)]$$

$$= [p^{\tau I}(\tau, \sigma), \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma_1 X^{J'}(\tau, \sigma_1) X^{J'}(\tau, \sigma_1)]$$

(from 12.15 — the other term in H commutes with $p^{\tau I}$)

$$[p^{\tau I}(\tau, \sigma), X^{J'}(\tau, \sigma_1)] = -i\eta^{IJ} \frac{d}{d\sigma_1} \delta(\sigma_1 - \sigma) \quad \text{so}$$

$$\frac{d p^{\tau I}(\tau, \sigma)}{d\tau} = -\frac{\eta^{IJ}}{2\pi\alpha'} \int_0^\pi d\sigma_1 X^{J'}(\tau, \sigma_1) \delta'(\sigma_1 - \sigma)$$

$$\left(\text{where } ' \text{ means } \frac{d}{d\sigma_1} \right) = +\frac{1}{2\pi\alpha'} X^{I''}(\tau, \sigma) \quad (\text{integrating by parts})$$

$$\text{So we have } \dot{X}^I = 2\pi\alpha' p^{\tau I} \quad \text{and} \quad \dot{p}^{\tau I} = \frac{1}{2\pi\alpha'} X^{I''}$$

$$\text{and so } \ddot{X}^I = X^{I''}$$

Problem 12.2 Testing explicitly some vanishing commutators

From 12.24, $X^{I'} = -i\sqrt{2\alpha'} \sum_{n \neq 0} \alpha_n^I \sin n\sigma e^{-in\tau}$

and $\dot{X}^I = \sqrt{2\alpha'} \left[\alpha_0^I + \sum_{n \neq 0} \alpha_n^I \cos n\sigma e^{-in\tau} \right]$

which we can rewrite as $X^{I'} = -i\sqrt{2\alpha'} \sum_n \alpha_n^I \sin n\sigma e^{-in\tau}$
 $\dot{X}^I = \sqrt{2\alpha'} \sum_n \alpha_n^I \cos n\sigma e^{-in\tau}$

(where now n goes over all integers). Since $[\alpha_m^I, \alpha_n^J] = m\eta_{IJ} \delta_{m+n,0}$

$$[X^{I'}(\tau, \sigma_1), X^{J'}(\tau, \sigma_2)] = +2\alpha' \sum_m m \sin m\sigma_1 \sin m\sigma_2 e^{-im\tau} e^{im\tau} \eta^{IJ}$$

$= 0$, since terms $\pm m$ cancel.

$$[\dot{X}^I(\tau, \sigma_1), \dot{X}^J(\tau, \sigma_2)] = 2\alpha' \sum_m m \cos m\sigma_1 \cos m\sigma_2 \eta^{IJ} = 0$$

(again $\pm m$ terms cancel).

[The sums do not converge, but we imagine multiplying by $f(\sigma_1)$, $g(\sigma_2)$ and integrating. If f and g are smooth, the sums will converge, and the $\pm m$ terms will cancel]

Problem 12.3 Testing explicitly the main commutator

$$a) X^I(\tau, \sigma_1) = x_0^I + \sqrt{2\alpha'} \left\{ \alpha_0^I \tau + i \sum_{m \neq 0} \frac{1}{m} \alpha_m^I \cos m\sigma_1 e^{-im\tau} \right\}$$

$$P^{IJ}(\tau, \sigma_2) = \frac{1}{\pi \sqrt{2\alpha'}} \left\{ \alpha_0^I + \sum_{n \neq 0} \alpha_n^I \cos n\sigma_2 e^{-in\tau} \right\}$$

$$\text{Since } [\alpha_m^I, \alpha_n^J] = \eta^{IJ} \delta_{m+n, 0} \text{ and } [x_0^I, \alpha_0^J] = i\sqrt{2\alpha'} \eta^{IJ}$$

$$\begin{aligned} [X^I(\tau, \sigma_1), P^{IJ}(\tau, \sigma_2)] &= i\eta^{IJ} \left\{ 1 + \sum_{m \neq 0} \cos m\sigma_1 \cos m\sigma_2 e^{-im\tau} e^{im\tau} \right\} \\ &= i\eta^{IJ} \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \cos n\sigma_1 \cos n\sigma_2 \end{aligned}$$

b) For any $f(\sigma)$, defined over $[0, \pi]$, we have $f(\sigma) = \sum_{n=0}^{\infty} A_n \cos n\sigma$

$$\int_0^{\pi} f(\sigma) \cos n\sigma d\sigma = \sum_{m=0}^{\infty} A_m \int_0^{\pi} \cos m\sigma \cos n\sigma d\sigma = A_n \int_0^{\pi} \cos^2 n\sigma d\sigma$$

$$= A_n \times \begin{cases} \frac{\pi}{2} & n=1, 2, 3, \dots \\ \pi & n=0 \end{cases}$$

$$\text{Hence } A_0 = \frac{1}{\pi} \int_0^{\pi} f(\sigma) d\sigma$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(\sigma) \cos n\sigma d\sigma \quad (n=1, 2, \dots)$$

$$\begin{aligned} \text{and } f(\sigma) &= \frac{1}{\pi} \int_0^{\pi} f(\sigma_1) d\sigma_1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \cos n\sigma \int_0^{\pi} f(\sigma_1) \cos n\sigma_1 d\sigma_1 \\ &= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \cos n\sigma \int_0^{\pi} \cos n\sigma_1 f(\sigma_1) d\sigma_1 \end{aligned}$$

$$\text{which says that } \delta(\sigma - \sigma_1) = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \cos n\sigma \cos n\sigma_1$$

$$\text{with } \sigma, \sigma_1 \in [0, \pi]$$

12.4 Analytic continuation of the zeta function.

Replacing the integration variable t with nt in the definition of $\Gamma(s)$ gives

$$\begin{aligned}\Gamma(s) &= \int_0^\infty d(nt) e^{-nt} n^{s-1} t^{s-1} = n^s \int_0^\infty dt e^{-nt} t^{s-1}. \\ \rightarrow \Gamma(s)\zeta(s) &= \sum_{n=1}^\infty \Gamma(s) n^{-s} = \sum_{n=1}^\infty \int_0^\infty dt e^{-nt} t^{s-1} = \int_0^\infty dt t^{s-1} \sum_{n=1}^\infty e^{-nt}.\end{aligned}$$

The geometric series gives $\sum_{n=1}^\infty e^{-nt} = \frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t-1}$, so we get

$$\Gamma(s)\zeta(s) = \int_0^\infty dt \frac{t^{s-1}}{e^t-1}. \quad (1)$$

We now consider the expansion of $1/(e^t-1)$ around $t=0$. Expanding the exponential,

$$\frac{1}{e^t-1} = \frac{1}{t + \frac{t^2}{2} + \frac{t^3}{3!} + \mathcal{O}(t^4)} = \frac{1}{t} \cdot \frac{1}{1 + \frac{t}{2} + \frac{t^2}{6} + \mathcal{O}(t^3)} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^2).$$

where we used $\frac{1}{1+x} = 1 - x + x^2 - \dots$. We now write $\Gamma(s)\zeta(s)$ in (1) as

$$\Gamma(s)\zeta(s) = \int_0^1 dt \frac{t^{s-1}}{e^t-1} + \int_1^\infty dt \frac{t^{s-1}}{e^t-1}. \quad (2)$$

Only the first integral on the right hand side may diverge near $t=0$. We rewrite it as

$$\begin{aligned}\int_0^1 dt \frac{t^{s-1}}{e^t-1} &= \int_0^1 dt t^{s-1} \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right) + \int_0^1 dt t^{s-1} \left(\frac{1}{t} - \frac{1}{2} + \frac{t}{12} \right) \\ &= \int_0^1 dt t^{s-1} \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right) + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)}.\end{aligned}$$

Hence, as quoted in the problem statement, (2) becomes

$$\Gamma(s)\zeta(s) = \int_0^1 dt t^{s-1} \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right) + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty dt \frac{t^{s-1}}{e^t-1}. \quad (3)$$

The last integral is convergent for all s . Since the terms enclosed by the large parentheses are of order t^2 , the first integral is of the form $\int_0^1 dt t^{s+1}$ and converges for $\Re(s) > -2$.

We recall that $\Gamma(s)$ has a simple pole at $s=0$ with residue one and a simple pole at $s=-1$ with residue -1 [the general result is that the residue at $s=-n < 0$, with n integer

is $(-1)^n/n!]$. Equation (3) shows that $\Gamma(s)\zeta(s)$ also has simple poles at $s = 0$ and $s = -1$. We therefore deduce that $\zeta(s)$ is regular for $s = 0, -1$, and

$$\text{Res}_{s=-n}(\Gamma(s)\zeta(s)) = (\text{Res}_{s=-n}\Gamma(s)) \zeta(-n), \quad n = 0, 1.$$

Reading off the residue at $s = -1$ from (3) we get the celebrated result:

$$\frac{1}{12} = (-1)\zeta(-1) \quad \rightarrow \quad \zeta(-1) = -\frac{1}{12}.$$

Similarly, reading off the residue at $s = 0$ we find that

$$-\frac{1}{2} = (1)\zeta(0) \quad \rightarrow \quad \zeta(0) = -\frac{1}{2}.$$

It is interesting to note in (3) that $\Gamma(s)\zeta(s)$ has a simple pole at $s = 1$. Since $\Gamma(s)$ has no pole at $s = 1$, $\zeta(s)$ does. The pole at $s = 1$, with residue one, is in fact the only singularity of the zeta function $\zeta(s)$.

Problem 12.5 The Virasoro algebra is a Lie algebra

$$[L_m, L_n] = (m-n)L_{m+n} \quad (\text{Virasoro algebra, no central extension})$$

$$[L_n, L_m] = (n-m)L_{n+m} = -(m-n)L_{m+n} = -[L_m, L_n]$$

$$[L_m, [L_n, L_p]] = (n-p)[L_m, L_{n+p}] \quad (\text{bracket is bilinear})$$

$$= (n-p)(m-n-p)L_{m+n+p} \quad \text{so Jacobi identity is satisfied}$$

$$\text{provided } \{m(n-p) - n^2 + p^2\} + \{n(p-m) - p^2 + m^2\} + \{p(m-n) - m^2 + n^2\} = 0$$

which is true.

$$\text{With central extension } [L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3-m)\delta_{m,-n}$$

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{1}{12}(n^3-n)\delta_{n,-m} = -[L_m, L_n]$$

(since last term only contributes when $m+n=0$ and changes sign when $m \rightarrow -m$)

$$[L_m, [L_n, L_p]] = (n-p)[L_m, L_{n+p}] \quad (\text{if we are thinking of this as an algebra, the central extension term should be thought of as a multiple of an element } 1 \text{ which has zero bracket } [L_m, 1] = 0)$$

$$= (n-p)(m-n-p)L_{m+n+p} + (n-p)\frac{1}{12}(m^3-m)\delta_{m+n+p,0}$$

So extra terms in Jacobi identity only are non-zero if $m+n+p=0$.

$$\text{In that case } \sum (n-p)m = 0 \quad \text{and} \quad \sum (n-p)m^3 = -\sum (n-p)(n+p)^3$$

$$= -\sum (n^2-p^2)(n^2+p^2+2np) = -\sum (n^4-p^4) + 2\sum mnp(n-p) = 0$$

($\sum$ means sum over 3 cyclic permutations of m, n, p)

Problem 12.6 Consistency conditions on the Virasoro anomaly

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0}$$

a) Antisymmetry requires $A(-m) = -A(m)$, so that in particular $A(0) = 0$

b) Exactly as in problem 12.5,

$$[L_m, [L_n, L_k]] = (n-k)(m-n-k)L_{m+n+k} + (n-k)A(n)\delta_{m+n+k,0}$$

so Jacobi identity requires $\sum (n-k)A(m) = 0$ when $m+n+k=0$

c) We already showed in 12.5 that $A(m) = m$ and $A(m) = m^3$, and so $A(m) = \alpha m + \beta m^3$ yield consistent central extensions

d) With $k=1$ and $n=-m-1$ so that $m+n+k=0$, we must have $(-m-2)A(m) + (1-m)A(-m-1) + (2m+1)A(1) = 0$

$$\text{i.e. } (m-1)A(m+1) = (m+2)A(m) - (2m+1)A(1)$$

For $m=2,3,\dots$ this gives $A(m+1) = \frac{m+2}{m-1}A(m) - \frac{(2m+1)}{m-1}A(1)$

so determines in turn $A(3), A(4), \dots$ in terms of $A(1)$ and $A(2)$.

[For $m=1$, equation gives $0 = 3A(1) - 3A(1)$]

All $A(m)$ are then determined in terms of $A(1), A(2)$ using antisymmetry.

[This tells us that $A(m) = \alpha m + \beta m^3$ is most general central extension, since $A(1) = \alpha + \beta$, $A(2) = 2\alpha + 8\beta$ can be solved for α, β]

Problem 12.7 Exercises with Virasoro operators

a) Suppose $L_1|\psi\rangle = 0, L_2|\psi\rangle = 0$

From (12.142), $[L_m, L_1] = (m-1)L_{m+1}, \text{ all } m \geq 2$

$$L_3|\psi\rangle = \frac{1}{2}(L_2L_1 - L_1L_2)|\psi\rangle = 0$$

$$L_4|\psi\rangle = \frac{1}{3}(L_3L_1 - L_1L_3)|\psi\rangle = 0 \quad \dots$$

(or use induction)

b) $[L_0, L_1] = -L_1 \quad [L_0, L_{-1}] = +L_{-1}$
 $[L_1, L_{-1}] = 2L_0$ (central term is zero for $m=1$)

So these 3 operators generate a subalgebra.

To calculate $L_n|0\rangle$ we use expressions for $L_n^\perp$ in terms of α_n

$$L_0^\perp = \sum_I \left\{ \frac{1}{2} \alpha_0^I \alpha_0^I + \alpha_{-1}^I \alpha_1^I + \alpha_{-2}^I \alpha_2^I + \dots \right\}$$

$$L_1^\perp = \sum_I \left\{ \alpha_0^I \alpha_1^I + \alpha_{-1}^I \alpha_2^I + \alpha_{-2}^I \alpha_3^I + \dots \right\}$$

$$L_{-1}^\perp = \sum_I \left\{ \alpha_{-1}^I \alpha_0^I + \alpha_{-2}^I \alpha_1^I + \alpha_{-3}^I \alpha_2^I + \dots \right\}$$

All these terms are normal ordered and $\alpha_n^I|0\rangle = 0, n \geq 0$

($\alpha_0^I = \sqrt{2\alpha'} p^I$ and $|0\rangle$ has zero momentum)

so we see that $L_0|0\rangle = L_1|0\rangle = L_{-1}|0\rangle = 0$

12.8 Virasoro operators acting on states.

(a) The idea is to split the infinite sum in $L_{-6}^\perp$ into terms with two creation operators (that can give something acting on the vacuum) and terms that are well-ordered and contain an annihilation operator (and therefore kill the vacuum). We have from (12.100) that

$$L_{-6}^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{-6+p}^I \alpha_{-p}^I,$$

where we changed p for $-p$ in the sum. We now split

$$L_{-6}^\perp = \frac{1}{2} \sum_{p=0}^6 \alpha_{-6+p}^I \alpha_{-p}^I + \frac{1}{2} \sum_{p=7}^{\infty} \alpha_{-6+p}^I \alpha_{-p}^I + \frac{1}{2} \sum_{p<0} \alpha_{-6+p}^I \alpha_{-p}^I.$$

Let $p = p' + 6$ in the second sum (so that p' then runs from one to infinity) and $p = -p'$ in the last sum (so that p' then runs from one to infinity):

$$L_{-6}^\perp = \frac{1}{2} \sum_{p=0}^6 \alpha_{-6+p}^I \alpha_{-p}^I + \frac{1}{2} \sum_{p'=1}^{\infty} \alpha_{p'}^I \alpha_{-6-p'}^I + \frac{1}{2} \sum_{p'=1}^{\infty} \alpha_{-6-p'}^I \alpha_{p'}^I.$$

Since the $[\alpha_{p'}, \alpha_{-6-p'}] = 0$ the second sum is equal to the third:

$$L_{-6}^\perp = \frac{1}{2} \sum_{p=0}^6 \alpha_{-6+p}^I \alpha_{-p}^I + \sum_{p=1}^{\infty} \alpha_{-6-p}^I \alpha_p^I.$$

The second sum contains one annihilator (the α_p^I) so it kills the zero-momentum vacuum $|0\rangle$. Expanding out the first sum we have

$$L_{-6}^\perp = \alpha_{-6}^I \alpha_0^I + \alpha_{-5}^I \alpha_{-1}^I + \alpha_{-4}^I \alpha_{-2}^I + \frac{1}{2} \alpha_{-3}^I \alpha_{-3}^I + \sum_{p=1}^{\infty} \alpha_{-6-p}^I \alpha_p^I.$$

Since α_0^I is a momentum operator it kills the zero momentum vacuum. We thus get

$$\boxed{L_{-6}^\perp |0\rangle = \left(\alpha_{-5}^I \alpha_{-1}^I + \alpha_{-4}^I \alpha_{-2}^I + \frac{1}{2} \alpha_{-3}^I \alpha_{-3}^I \right) |0\rangle.}$$

To evaluate $L_{-2}^\perp L_{-2}^\perp |0\rangle$ we first write out terms in $L_{-2}^\perp$:

$$L_{-2}^\perp = \frac{1}{2} \alpha_{-1}^I \alpha_{-1}^I + \left(\alpha_{-2}^I \alpha_0^I + \alpha_{-3}^I \alpha_1^I + \alpha_{-4}^I \alpha_2^I + \dots \right).$$

We then have

$$L_{-2}^{\perp}|0\rangle = \frac{1}{2}\alpha_{-1}^J\alpha_{-1}^J|0\rangle.$$

We now act with another $L_{-2}^{\perp}$:

$$L_{-2}^{\perp}L_{-2}^{\perp}|0\rangle = \left(\frac{1}{2}\underline{\alpha_{-1}^I\alpha_{-1}^I} + \alpha_{-2}^I\alpha_0^I + \underline{\alpha_{-3}^I\alpha_1^I} + \alpha_{-4}^I\alpha_2^I + \dots\right)\frac{1}{2}\alpha_{-1}^J\alpha_{-1}^J|0\rangle.$$

Only the underlined terms contribute. Noting that $[\alpha_1^I, \alpha_{-1}^J\alpha_{-1}^J] = 2\alpha_{-1}^I$, we get

$$L_{-2}^{\perp}L_{-2}^{\perp}|0\rangle = \left(\frac{1}{4}\alpha_{-1}^I\alpha_{-1}^I\alpha_{-1}^J\alpha_{-1}^J + \alpha_{-3}^I\alpha_{-1}^I\right)|0\rangle.$$

(b) This time we write out $L_2^{\perp}$

$$L_2^{\perp} = \frac{1}{2}\alpha_1^I\alpha_1^I + \left(\alpha_0^I\alpha_2^I + \alpha_{-1}^I\alpha_3^I + \alpha_{-2}^I\alpha_4^I + \dots\right).$$

Therefore,

$$L_2^{\perp}L_{-2}^{\perp}|0\rangle = \left(\frac{1}{2}\underline{\alpha_1^I\alpha_1^I} + \alpha_0^I\alpha_2^I + \alpha_{-1}^I\alpha_3^I + \alpha_{-2}^I\alpha_4^I + \dots\right)\frac{1}{2}\alpha_{-1}^J\alpha_{-1}^J|0\rangle.$$

Only the underlined term contributes, so we get

$$\begin{aligned} L_2^{\perp}L_{-2}^{\perp}|0\rangle &= \frac{1}{4}\alpha_1^I\alpha_1^I\alpha_{-1}^J\alpha_{-1}^J|0\rangle = \frac{1}{4}\alpha_1^I[\alpha_1^I, \alpha_{-1}^J\alpha_{-1}^J]|0\rangle \\ &= \frac{1}{2}\alpha_1^I\alpha_{-1}^I|0\rangle = \frac{1}{2}[\alpha_1^I, \alpha_{-1}^I]|0\rangle \\ &= \frac{1}{2}\eta^{II}|0\rangle = \frac{1}{2}(D-2)|0\rangle. \end{aligned}$$

Moreover, we have $L_2^{\perp}|0\rangle = 0$ and therefore $L_{-2}^{\perp}L_2^{\perp}|0\rangle = 0$. As a result, the above computation means that

$$\boxed{[L_2^{\perp}, L_{-2}^{\perp}]|0\rangle = \frac{1}{2}(D-2)|0\rangle.}$$

On the other hand, from the Virasoro algebra in (12.133) we have

$$[L_2^{\perp}, L_{-2}^{\perp}] = 4L_0^{\perp} + \frac{D-2}{12}(8-2) = 4L_0^{\perp} + \frac{1}{2}(D-2).$$

Since $L_0^{\perp}|0\rangle = 0$, letting the above act on the vacuum we recover the boxed equation.

Problem 12.8 Reparameterizations generated by Virasoro operators

a) We want the reparameterization to send $\tau=0$ to $\tau=0$, which means $\xi^\tau = 0$ when $\tau=0$. From (12.152) and (12.153) we see that $L_m^\perp - L_{-m}^\perp$ has the property ($m=1,2,3,\dots$).

$$[L_m - L_{-m}, L_n - L_{-n}] = (m-n)L_{m+n} + (-m+n)L_{-m-n} - (-m-n)L_{-m+n} - (m+n)L_{m-n}$$

(no contributions from central term since we only look at $m, n > 0$, $m \neq n$)

$$= (m-n)(L_{m+n} - L_{-(m+n)}) + (m+n)(L_{n-m} - L_{m-n})$$

If we call $(L_m - L_{-m}) = R_m$, $[R_m, R_n] = (m-n)R_{m+n} + (m+n)R_{n-m}$ and algebra closes i.e. the R_m form a subalgebra.
(If $m > n$, write $R_{n-m} = -R_{m-n}$)

b) The general reparameterization is generated by
$$\sum_{n=1}^{\infty} \left\{ u_n (L_n - L_{-n}) + v_n i (L_n + L_{-n}) \right\}$$

We want $\xi^\tau = 0$, $\xi^\sigma = 0$ when $\tau=0$, $\sigma = \frac{\pi}{2}$

From (12.152) and (12.153), this requires

$$\sum u_n \sin \frac{n\pi}{2} = 0 \quad \text{and} \quad \sum v_n \cos \frac{n\pi}{2} = 0$$

$$\text{i.e. } u_1 - u_3 + u_5 - \dots = 0$$

$$v_2 - v_4 + v_6 - \dots = 0$$

Problem 12.9 Reparameterizations and constraints

a) $\dot{\xi}_m^\tau = \dot{\xi}_m^\sigma / \dot{\xi}_m^\sigma = m e^{\text{limit}} \text{ as } m \rightarrow \infty, \quad \dot{\xi}_m^\sigma = \dot{\xi}_m^\tau / \dot{\xi}_m^\sigma = \lim_{m \rightarrow \infty} e^{\text{limit}} \text{ as } m \rightarrow \infty$

b) We can write constraints as $(\partial_\tau X \pm \partial_\sigma X)^2 = 0$

$$\frac{\partial X}{\partial \tau'} = \frac{\partial X}{\partial \tau} \frac{\partial \tau}{\partial \tau'} + \frac{\partial X}{\partial \sigma} \frac{\partial \sigma}{\partial \tau'} = \frac{\partial X}{\partial \tau} - \epsilon \left\{ \frac{\partial X}{\partial \tau} \frac{\partial \tau}{\partial \tau'} + \frac{\partial X}{\partial \sigma} \frac{\partial \sigma}{\partial \tau'} \right\}$$

$$\frac{\partial X}{\partial \sigma'} = \frac{\partial X}{\partial \tau} \frac{\partial \tau}{\partial \sigma'} + \frac{\partial X}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma'} = \frac{\partial X}{\partial \sigma} - \epsilon \left\{ \frac{\partial X}{\partial \tau} \frac{\partial \tau}{\partial \sigma'} + \frac{\partial X}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma'} \right\}$$

Since $\dot{\xi}_m^\tau = \dot{\xi}_m^\sigma / \dot{\xi}_m^\sigma$ and $\dot{\xi}_m^\sigma = \dot{\xi}_m^\tau / \dot{\xi}_m^\sigma$ we can write these as

$$\frac{\partial X}{\partial \tau'} \pm \frac{\partial X}{\partial \sigma'} = \frac{\partial X}{\partial \tau} \pm \frac{\partial X}{\partial \sigma} - \epsilon \left(\frac{\partial X}{\partial \tau} \pm \frac{\partial X}{\partial \sigma} \right) \left(\frac{\partial \tau}{\partial \tau'} \pm \frac{\partial \sigma}{\partial \tau'} \right)$$

$$= (1 - \epsilon \left(\frac{\partial \tau}{\partial \tau'} \pm \frac{\partial \sigma}{\partial \tau'} \right)) \left(\frac{\partial X}{\partial \tau} \pm \frac{\partial X}{\partial \sigma} \right)$$

$$\left(\frac{\partial X}{\partial \tau'} \pm \frac{\partial X}{\partial \sigma'} \right)^2 = (1 - 2\epsilon \left(\frac{\partial \tau}{\partial \tau'} \pm \frac{\partial \sigma}{\partial \tau'} \right)) \left(\frac{\partial X}{\partial \tau} \pm \frac{\partial X}{\partial \sigma} \right)^2 = 0.$$

(all to order ϵ)

12.11 Counting symmetric products.

We have a collection $\mathcal{C}$ of symbols

$$\mathcal{C} = \{ a^1, a^2, a^3, \dots, a^k \}. \quad (1)$$

We build products using n symbols. Any symbol in $\mathcal{C}$ can be used any number of times in forming the product. A product thus takes the form

$$a^{I_1} a^{I_2} a^{I_3} \dots a^{I_n}, \quad (2)$$

and we can easily order the symbols in the product so that

$$1 \leq I_1 \leq I_2 \leq I_3 \leq \dots \leq I_n \leq k. \quad (3)$$

Examples of products are $(a^1)^n$ or $(a^k)^n$ or $(a^1)^p(a^2)^q$ with $p + q = n$.

Think of representing each and every symbol in the collection $\mathcal{C}$ by a ball $\bullet$. Thus, a product as in (2) would use n balls:

$$\underbrace{\bullet \bullet \dots \bullet}_{n\text{-balls}} \quad (4)$$

This notation misses the information of the superscripts on the symbols since balls have no labels. These superscripts can take any of k values. To recover the information we use dividers $|$. It suffices to have $k - 1$ of them because placed along a line they create k separate spaces. Here are a few examples along the lines of the one in the problem statement, for which $k = 6$ and $n = 9$.

$$\begin{aligned} \bullet | \bullet \bullet | \bullet \bullet \bullet | \bullet | \bullet \bullet &\leftrightarrow a^1 (a^2)^2 (a^4)^3 a^5 (a^6)^2 \\ \bullet \bullet \bullet \bullet \bullet \bullet \bullet | | | | &\leftrightarrow (a^1)^9 \\ | | | | \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet &\leftrightarrow (a^6)^9 \\ \bullet \bullet \bullet | | | | \bullet \bullet \bullet \bullet \bullet &\leftrightarrow (a_1)^3 (a^6)^6 \end{aligned}$$

Any ordered configuration of the n balls and $k - 1$ dividers along a line corresponds to a uniquely specified product. Conversely, any product admits a unique representation in terms of balls and dividers. Thus the number N of products is equal to the number of inequivalent orderings, along a line, of the identical balls and the identical dividers.

The total number of objects, balls plus dividers, is $n + k - 1$. Thought all distinguishable they can be ordered along the line in $(n + k - 1)!$ ways. We must divide then by $n!$ indistinguishable reorderings of the balls and $(k - 1)!$ indistinguishable reorderings of the dividers. Thus, we find the claimed result

$$N(n, k) = \frac{(n + k - 1)!}{n! (k - 1)!}. \quad (5)$$

The quotient is nicely represented as the ratio of n ascending integers beginning with k divided by the n factors in $n!$:

$$N(n, k) = \frac{(n+k-1)!}{(k-1)!} \frac{1}{n!} = \frac{k(k+1)(k+2)\dots(k+n-1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}.$$

We can use the above result to count the number of states of the form

$$a_1^{I\dagger} a_1^{J\dagger} a_1^{K\dagger} |0\rangle$$

in open string theory. Here $n = 3$ (three oscillators) and $k = D - 2 = 24$ (24 values for the indices) so we get

$$N(3, 24) = \frac{24 \cdot 25 \cdot 26}{1 \cdot 2 \cdot 3} = 2600.$$

This confirms the statement below equation (12.181).

More generally, we can say that

$N(n, k)$ is the number of components in a rank n symmetric tensor in k dimensions.

This is the case because such tensor S has n indices, each of which can take k values. More explicitly the above claim follows because we can uniquely associate each component of the tensor to one of our products:

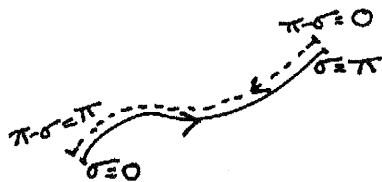
$$S_{I_1 I_2 \dots I_n} \leftrightarrow a^{I_1} a^{I_2} \dots a^{I_n}.$$

This means that the counting is the same.

Problem 12.10 Unoriented open strings

a) As σ goes from 0 to π , $\pi - \sigma$ goes from π to 0.

String $X^\mu(\tau, \pi - \sigma)$ covers the same set of space-time points X^μ as string $X^\mu(\tau, \sigma)$, but with end-points interchanged and orientation reversed.



b) $\cos(\pi - \sigma) = (-1)^n \cos \sigma$ so from (12.24) we see that $\Omega \alpha_0^I \Omega^{-1} = \alpha_0^I$, $\Omega \alpha_n^I \Omega^{-1} = (-1)^n \alpha_n^I$

c) The mode expansion of X^- is (12.95) and gives $\Omega \bar{X}(\tau, \sigma) \Omega^{-1} = \bar{X}(\tau, \pi - \sigma)$ provided $\Omega \alpha_0^- \Omega^{-1} = \alpha_0^-$ and $\Omega \alpha_n^- \Omega^{-1} = (-1)^n \alpha_n^-$

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}} \frac{1}{p^+} L_n^\perp = \frac{1}{\sqrt{2\alpha'}} \frac{1}{p^+} \frac{1}{2} \sum_p \alpha_{n-p}^I \alpha_p^I$$

$$\text{so if } \Omega p^+ \Omega^{-1} = p^+, \quad \Omega \alpha_n^- \Omega^{-1} = \frac{1}{\sqrt{2\alpha'}} \frac{1}{2p^+} \sum_p (-1)^{np} \alpha_{n-p}^I (-1)^p \alpha_p^I$$

$$= (-1)^n \alpha_n^- \text{ as required.}$$

$$d) |\lambda\rangle = \prod_{n,I} (\alpha_n^{I\dagger})^{\lambda_{n,I}} |p^+, \vec{p}^\perp\rangle$$

$$\Omega |\lambda\rangle = \prod_{n,I} (\Omega \alpha_n^{I\dagger} \Omega^{-1})^{\lambda_{n,I}} \Omega |p^+, \vec{p}^\perp\rangle = (-1)^{\sum_{n,I} \lambda_{n,I} n} |p^+, \vec{p}^\perp\rangle$$

$$= (-1)^{N_\lambda^\perp} |\lambda\rangle \quad (\text{see 12.180})$$

States with $N^\perp = 0$ are $|p\rangle$ with twist +1

States with $N^\perp = 1$ are $\sum_I \alpha_1^{I\dagger} |p\rangle$, twist -1

States with $N^\perp = 2$ are $\sum_I \alpha_2^{I\dagger} |p\rangle$ and $\sum_{IJ} \alpha_1^{I\dagger} \alpha_1^{J\dagger} |p\rangle$

($\sum_{IJ} = \sum_{JI}$), twist +1

States with $N^\perp = 3$ are $\sum_I \alpha_3^{I\dagger} |p\rangle$, $\sum_{IJ} \alpha_2^{I\dagger} \alpha_1^{J\dagger} |p\rangle$, $\sum_{IJK} \alpha_1^{I\dagger} \alpha_1^{J\dagger} \alpha_1^{K\dagger} |p\rangle$

($\sum_{IJK}$ symmetric), twist -1.

c) For a theory of unoriented strings, I would keep only states $|\psi\rangle$ with $\Omega|\psi\rangle = +|\psi\rangle$, and say that only operators $\mathcal{O}$ with $\Omega\mathcal{O}\Omega^{-1} = \mathcal{O}$ represent physical observables — which must include the Hamiltonian H , and in fact the space-time translation and Lorentz generators. This is a consistent option available in a quantum theory (unlike a classical theory). It is much more than saying Ω is a symmetry — it says that acting with Ω gives us back the same state. We just showed that $\Omega|\lambda\rangle = \pm|\lambda\rangle$ so that acting with Ω on a basis state $|\lambda\rangle$ gives us back the same state (the phase is irrelevant until we consider superpositions of states with $\Omega = +1$ and -1 . Ω acting on $|p\rangle + \alpha_1^+ |p\rangle$ gives $|p\rangle - \alpha_1^+ |p\rangle$ which is a different state.

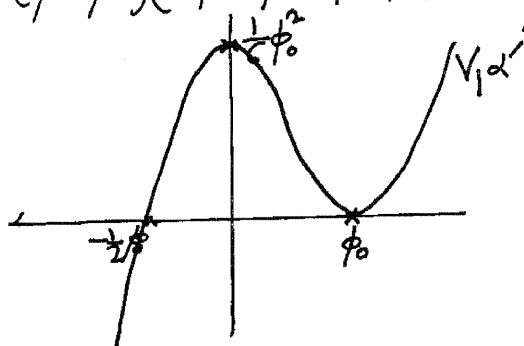
An analogous situation is the way permutation symmetry of identical particles appears in non-relativistic quantum mechanics. For a system of 2 α -particles all operators are symmetric but much more is true — only symmetric states exist. For 2 electrons only antisymmetric states exist. For open strings we could decide to keep only $\Omega = -1$ states, but then as soon as we tried to make strings interact we would run into severe constraints.

Problem 12.11 Tachyon potentials

a) and b)

$$\alpha' V_1(\phi) = \frac{1}{3\phi_0} (\phi - \phi_0)^2 (\phi + \frac{1}{2}\phi_0)$$

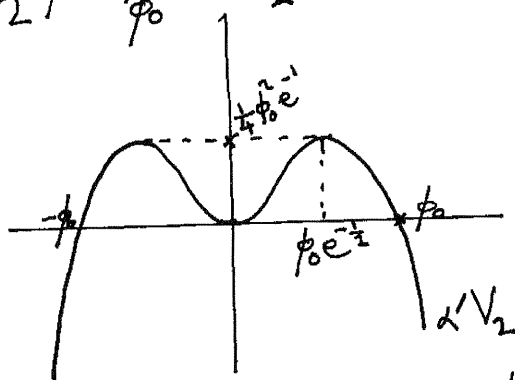
$$\alpha' V_1'(\phi) = \frac{1}{3\phi_0} (\phi - \phi_0)(2\phi + \phi_0 + \phi - \phi_0) = \frac{\phi}{\phi_0} (\phi - \phi_0)$$



Minimum at $\phi = \phi_0, V = 0$; maximum at $\phi = 0, V = \frac{1}{6}\phi_0^2$

$$\alpha' V_2(\phi) = -\frac{1}{4}\phi^2 \ln \frac{\phi^2}{\phi_0^2} = 0 \text{ at } \phi = 0, \phi = \pm\phi_0$$

$$\alpha' V_2'(\phi) = -\frac{1}{2}\phi \ln \frac{\phi^2}{\phi_0^2} - \frac{1}{2}\phi = 0 \text{ at } \phi = 0 \text{ and } \phi = \pm\phi_0 e^{-1/2}$$

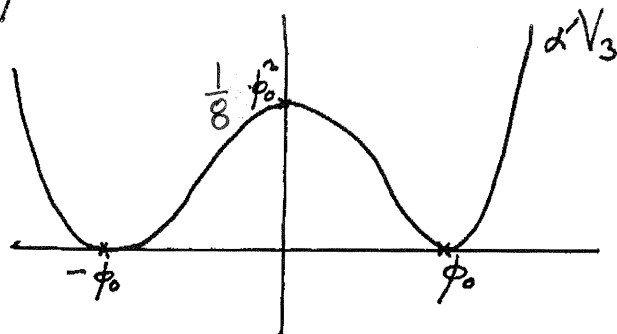


(at $\phi = 0, V_2''(\phi) = +\infty$)
but this doesn't show up
on my sketch

Minimum at $\phi = 0, V = 0$; maxima at $\phi = \pm\phi_0 e^{-1/2}, V = \frac{1}{4}\phi_0^2 e^{-1}$

$$\alpha' V_3(\phi) = \frac{1}{8} \frac{1}{\phi_0^2} (\phi^2 - \phi_0^2)^2$$

$$\alpha' V_3'(\phi) = \frac{1}{2\phi_0^2} \phi (\phi^2 - \phi_0^2)$$



Maximum at $\phi = 0$, $V = \frac{1}{8} \phi_0^2$

Minima at $\phi = \pm \phi_0$, $V = 0$

c) If $V'(\bar{\phi}) = 0$, $V(\bar{\phi} + \psi) \approx V(\bar{\phi}) + \frac{1}{2} V''(\bar{\phi}) \psi^2$

$$\alpha' V_1''(\phi) = 2\frac{\phi}{\phi_0} - 1 = +1 \text{ at } \phi = \phi_0, -1 \text{ at } \phi = 0$$

$$V_1(\phi_0 + \psi) \approx \frac{1}{2\alpha'} \psi^2 \quad ; \quad V_1(\psi) \approx \frac{1}{6\alpha'} \phi_0^2 - \frac{1}{2\alpha'} \psi^2$$

$$\alpha' V_2''(\phi) = -\frac{1}{2} \ln \frac{\phi^2}{\phi_0^2} - \frac{3}{2} = +\infty \text{ at } \phi = 0, -1 \text{ at } \phi = \pm \phi_0 e^{-\frac{1}{2}}$$

$V_2(\psi)$ does not have good Taylor series expansion at $\bar{\phi} = 0$, 'mass' = $+\infty$

$$V_2(\pm \phi_0 e^{-\frac{1}{2}} + \psi) \approx \frac{1}{4} \phi_0^2 e^{-1} - \frac{1}{2\alpha'} \psi^2$$

$$\alpha' V_3''(\phi) = \frac{3\phi^2}{2\phi_0^2} - \frac{1}{2} = -\frac{1}{2} \text{ at } \phi = 0, +1 \text{ at } \phi = \pm \phi_0$$

$$V_3(\psi) \approx \frac{1}{8\alpha'} \phi_0^2 - \frac{1}{4\alpha'} \psi^2 \quad ; \quad V_3(\pm \phi_0 + \psi) \approx \frac{1}{2\alpha'} \psi^2$$

Problem 13.1 Commutation relations for oscillators

a) From (13.28) $[(\dot{X}^I - X^{I'}) (\tau, \sigma), (\dot{X}^J - X^{J'}) (\tau, \sigma')] = -4\pi\alpha' i \eta^{IJ} \frac{d}{d\sigma} \delta(\sigma - \sigma')$

and from (13.26) $(\dot{X}^I - X^{I'}) (\tau, \sigma) = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha_n^I e^{-in(\tau - \sigma)}$

Operate with $\frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{in(\tau - \sigma)}$ on commutator.

$$[\sqrt{2\alpha'} \alpha_m^I, \sqrt{2\alpha'} \alpha_n^J] = -\frac{i\alpha'}{\pi} \eta^{IJ} \int_0^{2\pi} d\sigma \int_0^{2\pi} d\sigma' e^{i(mn)\tau - im\sigma - in\sigma'} \frac{d}{d\sigma} \delta(\sigma - \sigma')$$

Since $\int_0^{2\pi} f(\sigma) \frac{d}{d\sigma} \delta(\sigma - \sigma') d\sigma = -f'(\sigma')$ we get

$$[\alpha_m^I, \alpha_n^J] = -\frac{i}{2\pi} \eta^{IJ} \int_0^{2\pi} d\sigma' (+im) e^{i(mn)(\tau - \sigma')} = m \eta^{IJ} \delta_{m+n, 0}$$

b) Set of functions $e^{in\sigma}$ is complete, so any function $f(\sigma) = \sum_{n \in \mathbb{Z}} f_n e^{in\sigma}$

and $\int_0^{2\pi} f(\sigma) e^{-in\sigma} d\sigma = 2\pi f_n$ since $\int_0^{2\pi} e^{i(m-n)\sigma} d\sigma = 2\pi \delta_{m-n, 0}$

Hence $f(\sigma) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\sigma} \int_0^{2\pi} e^{-in\sigma'} f(\sigma') d\sigma' = \int_0^{2\pi} d\sigma' f(\sigma') \delta(\sigma - \sigma')$

and $\delta(\sigma - \sigma') = \frac{1}{2\pi} \sum_n e^{in(\sigma - \sigma')}$

c) From (13.24) $X^I(\tau, \sigma) = x_0^I + \sqrt{2\alpha'} \alpha_0^I \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (\alpha_n^I e^{in\sigma} + \bar{\alpha}_n^I e^{-in\sigma})$

From (13.20) $P^{IJ}(\tau, \sigma) = \frac{1}{2\pi\alpha'} \left\{ \sqrt{2\alpha'} \alpha_0^J + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} e^{-in\tau} (\alpha_n^J e^{in\sigma} + \bar{\alpha}_n^J e^{-in\sigma}) \right\}$

From (13.29) (13.30) and (13.33) we then get

$$[X^I(\tau, \sigma), P^{IJ}(\tau, \sigma')] = \frac{1}{\pi\sqrt{2\alpha'}} [\alpha_0^I, \alpha_0^J] + \frac{i\eta^{IJ}}{4\pi} \sum_{n \neq 0} \left\{ e^{in(\sigma - \sigma')} - e^{-in(\sigma - \sigma')} \right\}$$

$$= \frac{i\eta^{IJ}}{2\pi} \left\{ 1 + \sum_{n \neq 0} e^{in(\sigma - \sigma')} \right\} = i\eta^{IJ} \delta(\sigma - \sigma')$$

d) From (13.20) and (13.27)

$$[X^I(\tau, \sigma), \dot{X}^J(\tau, \sigma')] = 2\pi\alpha' i \eta^{IJ} \delta(\sigma - \sigma')$$

$\frac{1}{2\pi} \int_0^{2\pi} d\sigma$ removes all the terms in $e^{in\sigma}$, $n \neq 0$ from (13.24)

so we get $[x_0^I + \sqrt{2\alpha'} \alpha_0^I \tau, \dot{X}^J(\tau, \sigma')] = i \eta^{IJ} \alpha'$

Since the right hand side is independent of σ' , all the terms on the left in $e^{\pm in\sigma'}$, $n \neq 0$ must be zero so $[x_0^I + \sqrt{2\alpha'} \alpha_0^I \tau, \begin{Bmatrix} \alpha_n^J \\ \bar{\alpha}_n^J \end{Bmatrix}] = 0$

so x_0^I and α_0^I commute with $\alpha_n^J, \bar{\alpha}_n^J$, $n \neq 0$.

We are left with $[x_0^I + \sqrt{2\alpha'} \alpha_0^I \tau, \sqrt{2\alpha'} \alpha_0^J] = i \alpha' \eta^{IJ}$

so $[x_0^I, \alpha_0^J] = i \sqrt{\frac{\alpha'}{2}} \eta^{IJ}$ and $[\alpha_0^I, \alpha_0^J] = 0$

Problem 13.2 A projector into physical states

From (13.42) $P = L_0^\perp - \bar{L}_0^\perp = N^\perp - \bar{N}^\perp$ and the eigenvalues of $N^\perp$ and of $\bar{N}^\perp$ are $0, 1, 2, 3, \dots$ so the eigenvalues of $L_0^\perp - \bar{L}_0^\perp$ are $\dots -3, -2, -1, 0, 1, 2, \dots$

For any $|\lambda, \bar{\lambda}\rangle$, $P|\lambda, \bar{\lambda}\rangle = \{N^\perp(\lambda) - \bar{N}^\perp(\bar{\lambda})\} |\lambda, \bar{\lambda}\rangle$

so $e^{-iP\theta} |\lambda, \bar{\lambda}\rangle = e^{-i\{N^\perp_\lambda - \bar{N}^\perp_{\bar{\lambda}}\}\theta} |\lambda, \bar{\lambda}\rangle$

and $\mathcal{P}_0 |\lambda, \bar{\lambda}\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-iP\theta} |\lambda, \bar{\lambda}\rangle = \delta_{N^\perp_\lambda - \bar{N}^\perp_{\bar{\lambda}}, 0} |\lambda, \bar{\lambda}\rangle$

i.e. $\mathcal{P}_0 |\lambda, \bar{\lambda}\rangle = |\lambda, \bar{\lambda}\rangle$ if $|\lambda, \bar{\lambda}\rangle$ belongs to closed string state space
and $\mathcal{P}_0 |\lambda, \bar{\lambda}\rangle = 0$ ----- does not -----

and so $\mathcal{P}_0 |\psi\rangle \in \text{state space}$ for any $|\psi\rangle \in \mathcal{H}$

Problem 13.3 Action of $L_0^\perp - \bar{L}_0^\perp$

a) From (13.56) $i \frac{\partial X^I(\sigma)}{\partial \sigma} = [P, X^I(\sigma)]$

Think of this as a first order differential equation for $X^I(\sigma)$, which can be solved for $X^I(\sigma)$ given an initial value $X^I(\sigma_0)$.

Consider $F(\sigma) = e^{-iP(\sigma-\sigma_0)} X^I(\sigma_0) e^{iP(\sigma-\sigma_0)}$

$F(\sigma_0) = X^I(\sigma_0)$ and $i \frac{dF}{d\sigma} = e^{-iP(\sigma-\sigma_0)} [P, X^I(\sigma_0)] e^{iP(\sigma-\sigma_0)}$

$= [P, F(\sigma)]$

so $F(\sigma)$ is the required solution and must equal $X^I(\sigma)$

i.e. $e^{-iP(\sigma-\sigma_0)} X^I(\sigma_0) e^{iP(\sigma-\sigma_0)} = X^I(\sigma)$

which is the same as (13.59) (write $\sigma_0 \rightarrow \sigma$, $\sigma \rightarrow \sigma_0 + \sigma$)

(compare the equations for a Heisenberg operator $A(t)$ with H independent of t)

b) With $e^{-iP\sigma_0} X^I(\tau, \sigma) e^{iP\sigma_0} = X^I(\tau, \sigma + \sigma_0)$, differentiate with respect to τ and σ .

$e^{-iP\sigma_0} \dot{X}^I(\tau, \sigma) e^{iP\sigma_0} = \dot{X}^I(\tau, \sigma + \sigma_0)$ and

$e^{-iP\sigma_0} X^{I'}(\tau, \sigma) e^{iP\sigma_0} = \frac{d}{d\sigma} X^I(\tau, \sigma + \sigma_0) = X^{I'}(\tau, \sigma + \sigma_0)$

c) From (13.26) and (b),

$e^{-iP\sigma_0} \left\{ \sum_n \bar{a}_n^I e^{-in\sigma} \right\} e^{iP\sigma_0} = \sum_n \bar{a}_n^I e^{-in(\sigma+\sigma_0)}$

so $e^{-iP\sigma_0} \bar{a}_n^I e^{iP\sigma_0} = \bar{a}_n^I e^{-in\sigma_0}$

$e^{-iP\sigma_0} \left\{ \sum_n a_n^I e^{+in\sigma} \right\} e^{iP\sigma_0} = \sum_n a_n^I e^{in(\sigma+\sigma_0)}$

so $e^{-iP\sigma_0} a_n^I e^{iP\sigma_0} = a_n^I e^{in\sigma_0}$

$$d) \quad |U\rangle = \alpha_{-m}^I \bar{\alpha}_{-n}^J |P\rangle$$

$$e^{-iP_0\sigma} |U\rangle = (e^{-iP_0\sigma} \alpha_{-m}^I e^{iP_0\sigma}) (e^{-iP_0\sigma} \bar{\alpha}_{-n}^J e^{iP_0\sigma}) (e^{-iP_0\sigma} |P\rangle)$$

Using $P|P\rangle = 0$ so that $e^{-iP_0\sigma} |P\rangle = |P\rangle$, and (c),

$$e^{-iP_0\sigma} |U\rangle = e^{-im\sigma} e^{in\sigma} |U\rangle$$

so $|U\rangle$ is invariant under σ translations if $m=n$

(which is of course the condition that $P|U\rangle = 0$)

Problem 13.4

a) From (13.36) $4\alpha' \left\{ \begin{matrix} L_0^\perp \\ \bar{L}_0^\perp \end{matrix} \right\} = \frac{1}{2\pi} \int_0^{2\pi} (\dot{X}^\perp \mp X'^\perp)^2 d\sigma$

so $L_0^\perp - \bar{L}_0^\perp = -\frac{1}{2\pi\alpha'} \int_0^{2\pi} \dot{X}^\perp X'^\perp d\sigma$

[The ordering problems in defining $\dot{X}^\perp X'^\perp$ in the quantum theory are discussed in section 13.2]

13.4 $L_0^\perp - \bar{L}_0^\perp$ as world-sheet momentum. (Continued)

(b) With $\delta X^I = \epsilon \partial_\sigma X^I$ we find:

$$\delta \mathcal{L} = \frac{1}{2\pi\alpha'} \epsilon \left(\partial_\tau X^I \partial_\tau \partial_\sigma X^I - \partial_\sigma X^I \partial_\sigma^2 X^I \right) = \frac{1}{4\pi\alpha'} \epsilon \partial_\sigma \left((\partial_\tau X^I)^2 - (\partial_\sigma X^I)^2 \right).$$

With a little rearrangement we write

$$\delta \mathcal{L} = \partial_\sigma \left[\epsilon \frac{1}{4\pi\alpha'} \left((\partial_\tau X^I)^2 - (\partial_\sigma X^I)^2 \right) \right].$$

Since the variation of $\mathcal{L}$ is a total derivative, the σ -translation $\delta X^I = \epsilon \partial_\sigma X^I$ is a symmetry. In the notation of equation (1) of Problem 8.7, and letting $(\xi^0, \xi^1) = (\tau, \sigma)$ we write

$$\delta \mathcal{L} = \frac{\partial}{\partial \xi^\alpha} (\epsilon \Lambda^\alpha) \quad \text{with} \quad \Lambda^0 = 0 \quad \text{and} \quad \Lambda^1 = \frac{1}{4\pi\alpha'} \epsilon \left((\partial_\tau X^I)^2 - (\partial_\sigma X^I)^2 \right). \quad (1)$$

The conserved current is (equation (2) in Problem 8.7):

$$\epsilon j^\alpha = \frac{\partial \mathcal{L}}{\partial (\partial_\alpha X^I)} \delta X^I - \epsilon \Lambda^\alpha,$$

so the charge density j^0 is

$$j^0 = \frac{\partial \mathcal{L}}{\partial (\partial_0 X^I)} \partial_\sigma X^I - \Lambda^0 = \frac{1}{2\pi\alpha'} \dot{X}^I X^{I'}.$$

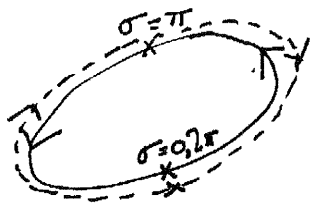
Note that Λ^α , computed to prove that we had a symmetry, does not enter into the charge density since $\Lambda^0 = 0$. The conserved charge Q is

$$Q = \int_0^{2\pi} d\sigma j^0 = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \dot{X}^I X^{I'} = -(L_0^\perp - \bar{L}_0^\perp),$$

comparing with the result stated in part (a) of the problem.

Problem 13.5 Unoriented closed strings

a)



Same set of points traversed from $X(\sigma=0)$ in opposite directions, returning to $X(\sigma=0)$.

b) Since $e^{\pm i n (2\pi - \sigma)} = e^{\mp i n \sigma}$, from (13.24) we get

$$\Omega \alpha_0^I \Omega^{-1} = \alpha_0^I, \quad \Omega \alpha_n^I \Omega^{-1} = \alpha_n^I$$

$$\Omega \bar{\alpha}_n^I \Omega^{-1} = \bar{\alpha}_n^I, \quad \Omega \bar{\alpha}_n^I \Omega^{-1} = \bar{\alpha}_n^I$$

c) From (b), $\Omega L_n^\perp \Omega^{-1} = \bar{L}_n^\perp, \quad \Omega \bar{L}_n^\perp \Omega^{-1} = L_n^\perp$

and then from (13.39) and (13.40)

$$\Omega (\dot{X}^\pm(\sigma) \pm X'^\pm(\sigma)) \Omega^{-1} = \dot{X}^\pm(2\pi - \sigma) \mp X'^\pm(2\pi - \sigma)$$

Since we assume $\Omega \alpha_0^- \Omega^{-1} = \alpha_0^-$ this gives

$$\Omega X^-(\sigma) \Omega^{-1} = X^-(2\pi - \sigma)$$

d) We want states with $N^\perp = \bar{N}^\perp \leq 2$

i) $N^\perp = \bar{N}^\perp = 0, \quad |p\rangle, \quad \Omega = +1$

ii) $N^\perp = \bar{N}^\perp = 1, \quad \sum_{I,J} R_{IJ} a_I^{\dagger} \bar{a}_J^{\dagger} |p\rangle$

Action of Ω interchanges a with $\bar{a}$ so changes R_{IJ} to R_{JI}

So if we write $R_{IJ} = S_{IJ} + A_{IJ}$ as in (13.65)

the S_{IJ} states (graviton + dilation) have $\Omega = +1$
 and the A_{IJ} states (Kalb-Ramond) have $\Omega = -1$

iii) $N^\perp = \bar{N}^\perp = 2$

States are $\sum_{I,J} R_{IJ} a_2^{I\dagger} \bar{a}_2^{J\dagger} |p\rangle$ which break up into
 $\frac{1}{2}(D-2)(D-1)$ states with $\Omega = +1$ and $\frac{1}{2}(D-2)(D-3)$ states with $\Omega = -1$

and $\sum_{I,J,K} R_{I,JK} a_2^{I\dagger} \bar{a}_1^{J\dagger} \bar{a}_1^{K\dagger} |p\rangle + \bar{R}_{I,JK} \bar{a}_2^{I\dagger} a_1^{J\dagger} a_1^{K\dagger} |p\rangle$

where $R_{I,JK}$ and $\bar{R}_{I,JK}$ are each symmetric in J, K .

Action of Ω interchanges R and $\bar{R}$ so by taking $\bar{R}_{I,JK} = \pm R_{I,JK}$
 we get $(D-2)\frac{1}{2}(D-2)(D-1)$ states with $\Omega = +1$ and with $\Omega = -1$

and $\sum_{I,J,K,L} R_{IJ,KL} a_1^{I\dagger} a_1^{J\dagger} \bar{a}_1^{K\dagger} \bar{a}_1^{L\dagger} |p\rangle$

where R is symmetric under interchange of I with J , and also symmetric
 under interchange of K with L . Ω interchanges IJ with KL .

Think of (IJ) and (KL) as a single index that takes $\frac{1}{2}(D-2)(D-1)$ values,

and write $R_{(IJ)(KL)} = S_{(IJ)(KL)} + A_{(IJ)(KL)}$, the sum

of a symmetric and an antisymmetric matrix. Thus there are
 $\frac{1}{2}E(E+1)$ states with $\Omega = +1$ and $\frac{1}{2}E(E-1)$ with $\Omega = -1$

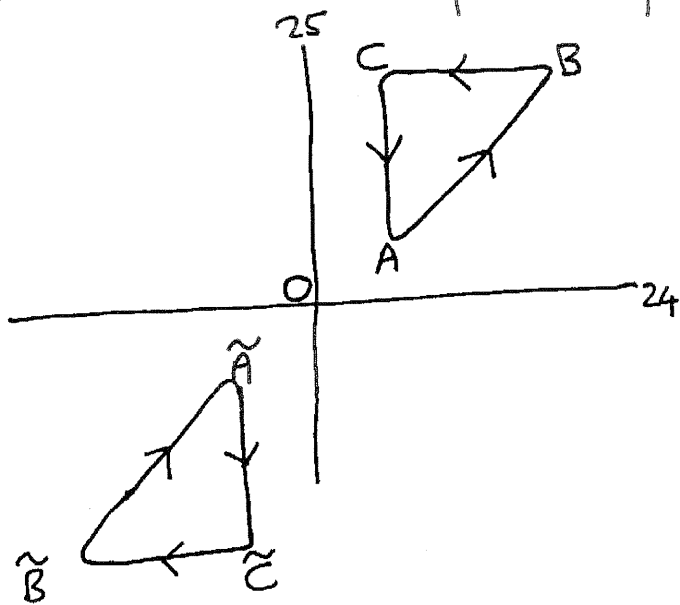
where $E = \frac{1}{2}(D-2)(D-1)$.

For unoriented strings, keep only states with $\Omega = +1$

Massless fields are then graviton and dilaton only.

Problem 13.6 Orientifold O_p -planes

a)



$\tilde{A} \tilde{B} \tilde{C}$ are
reflexions of ABC
in O .

Original string is traversed
in direction $ABCA$
Transformed string is traversed
in direction $\tilde{A} \tilde{C} \tilde{B} \tilde{A}$

b) Changing σ to $2\pi - \sigma$ interchanges α_n and $\bar{\alpha}_n$
(since $\alpha_0 = \bar{\alpha}_0$ there is no effect on α_0 , and none on x_0)
In addition Ω_p changes sign of all coefficients in expansion of X^a .
So $x_0^a \rightarrow -x_0^a$, $p^a \rightarrow -p^a$, $\alpha_n^a \rightarrow -\bar{\alpha}_n^a$, $\bar{\alpha}_n^a \rightarrow -\alpha_n^a$,
 $x_0^i \rightarrow x_0^i$, $p^i \rightarrow p^i$, $\alpha_n^i \rightarrow \bar{\alpha}_n^i$, $\bar{\alpha}_n^i \rightarrow \alpha_n^i$

Since $X^+ = \alpha' p^+ \tau$, $X^+(\tau, \sigma) \rightarrow X^+(\tau, \sigma) = X^+(\tau, 2\pi - \sigma)$

Since $X^-(\tau, \sigma)$ has mode expansion with coefficients $\bar{x}_0$, $\bar{L}_n$, $\bar{L}_n^a$
Each term in $\bar{L}_n$ is quadratic in α_r^I so the reflection of the α_r^I
does not affect $\bar{L}_n$. The change of orientation interchanges $\bar{L}_n^{\pm}$, $\bar{L}_n^{\pm}$
so changes σ to $2\pi - \sigma$, so $X^-(\tau, \sigma) \rightarrow X^-(\tau, 2\pi - \sigma)$

$$H = L_0^{\perp} + \bar{L}_0^{\perp} - 2 \quad (13.49) \quad \text{so } \Omega_p H \Omega_p^{-1} = H,$$

which means Ω_p is a symmetry of closed string theory.

c) Whenever we have an operator B with eigenstates $|b\rangle$,
 $B|b\rangle = b|b\rangle$, and $\Omega B \Omega^{-1} = \pm B$,

then $B\Omega|b\rangle = \pm \Omega B|b\rangle = \pm b\Omega|b\rangle$ so $\Omega|b\rangle$
 is an eigenstate of B with eigenvalue $\pm b$.

The state $|p^+, p^i, p^a\rangle$ satisfies $a_n^I |p\rangle = 0$.

so $a_n^I \Omega |p\rangle = \bar{a}_n^I |p\rangle = 0$. Similarly $\bar{a}_n^I \Omega |p\rangle = 0$

Since $\Omega p^+ \Omega^{-1} = p^+$, $\Omega p^i \Omega^{-1} = p^i$, $\Omega p^a \Omega^{-1} = -p^a$,

$\Omega |p^+, p^i, p^a\rangle$ must be a ground state with momenta $p^+, p^i, -p^a$

$$\text{i.e. } \Omega |p^+, p^i, p^a\rangle = e^{i\theta(p)} |p^+, p^i, -p^a\rangle$$

The phase is (almost) determined by the requirement that
 x_0^I and p^I are represented in the standard way as canonical conjugates
 i.e. that $x_0^I = i \frac{\partial}{\partial p^I}$. This only works if $\theta(p)$ is independent

of p , and since we are told $e^{i\theta} = +1$ when $p^a = 0$,

$$\Omega |p^+, p^i, p^a\rangle = |p^+, p^i, -p^a\rangle$$

d) Ω_p interchanges a_n^I with $\bar{a}_n^I$ and changes sign of $a_n^a, \bar{a}_n^a$
 Interchange leaves Φ_{IJ}^+ unchanged but changes sign of Φ_{IJ}^-

$$\text{Hence } \Phi_{ab}^\pm(p^+, p^i, p^a) = \pm \Phi_{ab}^\pm(p^+, p^i, -p^a)$$

$$\Phi_{ia}^\pm(p^+, p^i, p^a) = \mp \Phi_{ia}^\pm(p^+, p^i, -p^a)$$

$$\Phi_{ij}^\pm(p^+, p^i, p^a) = \pm \Phi_{ij}^\pm(p^+, p^i, -p^a)$$

gives states with $\Omega = +1$

Chapter 14 Selected Quick Calculations.

QC14.4. We look at the last line of (14.38).

$\alpha_{-1}^I b_{-1/2}^J$ on the vacuum gives $8 \times 8 = 64$ states.

$b_{-3/2}^I$ on the vacuum gives 8 states.

$b_{-1/2}^I b_{-1/2}^J b_{-1/2}^K$ gives $\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$ states.

We thus got a total of 128 states.

QC14.6. With $\alpha' M^2 = 2$ we have $N^\perp = 3$ so the states are

α_{-3}^I on the vacuum gives 24 states.

$\alpha_{-2}^I \alpha_{-1}^J$ on the vacuum gives $24 \times 24 = 576$ states.

$\alpha_{-1}^I \alpha_{-1}^J \alpha_{-1}^K$ gives $\frac{24 \cdot 25 \cdot 26}{1 \cdot 2 \cdot 3} = 2600$ states.

We thus got a total of $24 + 576 + 2600 = 3200$ states.

QC14.7. Now $D = 10$ so there are 8 transverse dimensions.

A graviton is a symmetric traceless transverse tensor, so we get $\frac{8 \times 9}{2} - 1 = 35$ states.
You can also use (10.108).

The Kalb-Ramond field is an antisymmetric transverse tensor, so we get $\frac{8 \times 7}{2} = 28$ states.

The dilaton counts for 1 state.

We thus got a total of $35 + 28 + 1 = 64$ states.

QC14.8. We can certainly combine NS- with NS- to form an closed string (NS-, NS-) sector. Both the left and right states appear at the same mass levels so that we can form consistent closed string states. We cannot combine NS- with R+ or R- because NS- has states at fractional $\alpha' M^2$ while the states in the Ramond sector (+ or -) occur at integer $\alpha' M^2$. Once cannot satisfy level matching.

Problem 14.1: Counting bosonic states

- (a) How many products N of the form $a^{i_1} a^{i_2}$ can be built? The answer is the same as the number of independent components of a symmetric matrix $M_{i_1 i_2}$, since the general operator of this form can be written as $M_{i_1 i_2} a^{i_1} a^{i_2}$. For $i_1 = i_2$, there are k possibilities. For $i_1 \neq i_2$, there are $k(k-1)/2$ possibilities, since i_1 can be chosen from k possibilities, i_2 can be chosen from the remaining $k-1$ possibilities, and the answer is divided by 2 to correct for the fact that each pair would otherwise be counted twice, once in each order. Adding the two cases,

$$N = \frac{k(k+1)}{2} ,$$

which is a formula that you may very well have already known. This result also arises from Problem 12.11 which gives

$$N(n, k) = \frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n} \quad (1)$$

for products with n elements each of which can be of k types.

To count the number of $a^{i_1} a^{i_2}$ we have $n = 2$ in eq.(1) and we get the earlier result:

$$N = \frac{k(k+1)}{1 \cdot 2} ,$$

How many $a^{i_1} a^{i_2} a^{i_3}$? This time $n = 3$ so we get

$$N = \frac{k(k+1)(k+2)}{6}$$

How many $a^{i_1} a^{i_2} a^{i_3} a^{i_4}$?

$$N = \frac{k(k+1)(k+2)(k+3)}{24} .$$

- (b) For the bosonic open string, the generating function is given as Eq. (14.62):

$$\begin{aligned} f_{os}(x) &= \frac{1}{x} \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{24}} \\ &= \frac{1}{x} + 24 + 324x + 3200x^2 + 25650x^3 + 176256x^4 + \dots \end{aligned}$$

The number of states with $\alpha' M^2 = 3$ is 25,650, the coefficient of x^3 . To enumerate them, we recall that

$$M^2 = \frac{1}{\alpha'} (N^\perp - 1) ,$$

so $\alpha' M^2 = 3$ implies that $N^\perp = 4$. Using only $n = 1$ creation operators, the states have the form

$$a_1^{I\dagger} a_1^{J\dagger} a_1^{K\dagger} a_1^{L\dagger} |p^+ \vec{p}_T\rangle ,$$

which includes $N_{1,1,1,1}$ independent states, where, according to Eq. (1),

$$N_{1,1,1,1} = \frac{24 \cdot 25 \cdot 26 \cdot 27}{24} = 25 \cdot 26 \cdot 27 = 17,550 .$$

States can also be built by one $n = 2$ creation operator and two $n = 1$ creation operators,

$$a_2^{I\dagger} a_1^{J\dagger} a_1^{K\dagger} |p^+ \vec{p}_T\rangle ,$$

which gives

$$N_{2,1,1} = 24 \times \frac{24 \cdot 25}{2} = 7,200$$

independent states. Using two $n = 2$ creation operators,

$$a_2^{I\dagger} a_2^{J\dagger} |p^+ \vec{p}_T\rangle ,$$

one can construct

$$N_{2,2} = \frac{24 \cdot 25}{2} = 300$$

states. Alternatively, one can use one $n = 3$ creation operator and one $n = 1$ creation operator,

$$a_3^{I\dagger} a_1^{J\dagger} |p^+ \vec{p}_T\rangle ,$$

which allows

$$N_{3,1} = 24 \times 24 = 576$$

options. Finally, one can use a single $n = 4$ operator,

$$a_4^{I\dagger} |p^+ \vec{p}_T\rangle ,$$

for which there are

$$N_4 = 24$$

possibilities. Adding them, one finds a total of 25,650 states, exactly as expected.

Problem 14.2: Generating function for the unoriented bosonic open string theory

The unoriented open string theory was discussed in Problem 12.12 of the textbook. In the solution to that problem, we learned that for bosonic open strings, the orientation-reversing twist operator Ω can be written

$$\Omega = (-1)^{N^\perp} .$$

The unoriented open bosonic string theory then includes all states for which $\Omega = 1$, which means all states for which $N^\perp$ is even. Since the energy level $\alpha' M^2 = N^\perp - 1$, the energy levels that we keep must be odd, starting with -1 . The full open bosonic string theory generating function is given by

$$\begin{aligned} f_{os}(x) &= \frac{1}{x} \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{24}} \\ &= \frac{1}{x} + 24 + 324x + 3200x^2 + 25650x^3 + 176256x^4 + \dots \end{aligned}$$

Recall that the power of x is the value of the energy level. We want to construct the series that eliminates the even-powered terms:

$$f_{os, \text{unoriented}} = \frac{1}{x} + 324x + 25650x^3 + \dots$$

The even and odd-powered terms in $f_{os}(x)$ can easily be distinguished by considering

$$f_{os}(-x) = -\frac{1}{x} + 24 - 324x + 3200x^2 - 25650x^3 + 176256x^4 + \dots ,$$

so one can see that

$$\begin{aligned} f_{os, \text{unoriented}} &= \frac{f_{os}(x) - f_{os}(-x)}{2} \\ &= \frac{1}{2x} \left\{ \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{24}} + \prod_{n=1}^{\infty} \frac{1}{(1-(-x)^n)^{24}} \right\} . \end{aligned}$$

Problem 14.3: Massive level in the open superstring

- (a) Given eight anticommuting variables b^i , with $i = 1, \dots, 8$, the number of inequivalent products that one can construct by multiplying n of them is easily calculated. First, of course, we need to be clear about what *inequivalent* means. If the ordering of two b 's is interchanged, the value of the product will be multiplied by -1. The new quantity is not equal to the original, but we will consider it equivalent in the sense that it is linearly dependent. Our primary interest will be the space of states that can be constructed by applying n anticommuting creation operators to some ground state, and the dimension of this space is the number of linearly independent states that can be constructed.

The number of linearly independent quantities that can be constructed by multiplying n b^i 's is simply

$$N = \frac{8 \cdot 7 \cdot 6 \dots (8 - n + 1)}{n!} = \frac{8!}{n! (8 - n)!} ,$$

which is the familiar binomial coefficient “8 choose n ”. If there were m anticommuting variables, it would be “ m choose n ”. (To understand the answer, first note that the product of anticommuting variables will vanish if any two indices are equal, so they must all be different. Thus first index can take any of 8 values, the second index can take any of the remaining 7 values, etc., and then the result must be divided by $n!$ to compensate for the $n!$ different orderings of the n distinct indices.)

How many $b^{i_1} b^{i_2}$? $N = \frac{8!}{2!6!} = 28$.

How many $b^{i_1} b^{i_2} b^{i_3}$? $N = \frac{8!}{3!5!} = 56$.

How many $b^{i_1} b^{i_2} b^{i_3} b^{i_4}$? $N = \frac{8!}{4!4!} = 70$.

- (b) For the NS sector of the open superstring, M^2 is given by Eq. (14.37):

$$M^2 = \frac{1}{\alpha'} \left(N^\perp - \frac{1}{2} \right) , \quad \text{where} \quad N^\perp = \sum_{p=1}^{\infty} \alpha_{-p}^I \alpha_p^I + \sum_{r=\frac{1}{2}, \frac{3}{2}, \dots} r b_{-r}^I b_r^I .$$

For $\alpha' M^2 = 1$, $N^\perp = 3/2$, which can be constructed in 3 ways:

- i) $b_{-\frac{1}{2}}^I b_{-\frac{1}{2}}^J b_{-\frac{1}{2}}^K |\text{NS}\rangle$: From part (a), $N = 56$.
- ii) $b_{-\frac{3}{2}}^I |\text{NS}\rangle$: $N = 8$.
- iii) $\alpha_{-1}^I b_{-\frac{1}{2}}^J |\text{NS}\rangle$: $N = 8 \times 8 = 64$.

The total for $\alpha' M^2 = 1$ is then 128. For $\alpha' M^2 = 2$, we have $N^\perp = 5/2$, which can be constructed in 7 ways:

- i) $\alpha_{-2}^I b_{-\frac{1}{2}}^J |\text{NS}\rangle$: $N = 8 \times 8 = 64$.

- ii) $\alpha_{-1}^I \alpha_{-1}^J b_{-\frac{1}{2}}^K |\text{NS}\rangle$: $N = \frac{8 \cdot 9}{2} \times 8 = 288$.
- iii) $\alpha_{-1}^I b_{-\frac{1}{2}}^J b_{-\frac{1}{2}}^K b_{-\frac{1}{2}}^L |\text{NS}\rangle$: $N = 8 \times 56 = 448$.
- iv) $\alpha_{-1}^I b_{-\frac{3}{2}}^J |\text{NS}\rangle$: $N = 8 \times 8 = 64$.
- v) $b_{-\frac{1}{2}}^I b_{-\frac{1}{2}}^J b_{-\frac{1}{2}}^K b_{-\frac{1}{2}}^L b_{-\frac{1}{2}}^M |\text{NS}\rangle$: $N = \frac{8!}{5!3!} = 56$.
- vi) $b_{-\frac{3}{2}}^I b_{-\frac{1}{2}}^J b_{-\frac{1}{2}}^K |\text{NS}\rangle$: $N = 8 \times 28 = 224$.
- vii) $b_{-\frac{5}{2}}^I |\text{NS}\rangle$: $N = 8$.

The total for $\alpha' M^2 = 1$ is then 1152.

For the $R-$ (fermionic) Ramond sector, M^2 is given by Eq. (14.53):

$$M^2 = \frac{1}{\alpha'} N^\perp, \quad \text{where} \quad N^\perp = \sum_{n=1}^{\infty} (\alpha_{-n}^I \alpha_n^I + n d_{-n}^I d_n^I) .$$

For $\alpha' M^2 = 1$, the states are already listed for us on the left side of Eq. (14.57). There are just two ways to construct such states:

- i) $\alpha_{-1}^I |R_a\rangle$: $N = 8 \times 8 = 64$. (Note that there are 8 states $|R_a\rangle$.)
- ii) $d_{-1}^I |R_{\bar{a}}\rangle$: $N = 8 \times 8 = 64$.

Thus the total is 128, in agreement with the Neveu-Schwarz sector for $\alpha' M^2 = 1$.

For $\alpha' M^2 = 2$, again the states are listed on the left side of Eq. (14.54):

- i) $\alpha_{-2}^I |R_a\rangle$: $N = 8 \times 8 = 64$.
- ii) $\alpha_{-1}^I \alpha_{-1}^J |R_a\rangle$: $N = \frac{8 \cdot 9}{2} \times 8 = 288$.
- iii) $\alpha_{-1}^I d_{-1}^J |R_{\bar{a}}\rangle$: $N = 8 \times 8 \times 8 = 512$.
- iv) $d_{-1}^I d_{-1}^J |R_a\rangle$: $N = 28 \times 8 = 224$.
- v) $d_{-2}^I |R_{\bar{a}}\rangle$: $N = 8 \times 8 = 64$.

The total is 1152, as expected.

Problem 14.4: Closed string degeneracies

The mass formula for closed bosonic strings is given in the textbook as Eq. (13.63), which can be written as

$$\frac{1}{2}\alpha' M^2 = (N^\perp - 1) + (\bar{N}^\perp - 1) \quad (1a)$$

$$= \alpha' M^2|_R + \alpha' M^2|_L, \quad (1b)$$

where R and L refer to the right-moving and left-moving excitations, respectively, and we must satisfy the constraint

$$\alpha' M^2|_R = \alpha' M^2|_L. \quad (2)$$

The constraint could of course be used to simplify Eq. (1), but doing so would spoil the manifest L - R symmetry. The M^2 formula for superstrings can be written as Eq. (1b), where $\alpha' M^2|_R$ and $\alpha' M^2|_L$ are interpreted as the values of $\alpha' M^2$ for the right-moving and left-moving modes, where each is evaluated as it would be for the open superstring.

- (a) For the closed bosonic string, the first five allowed values of $\alpha' M^2|_R = N^\perp - 1$ are -1, 0, 1, 2, and 3. The corresponding degeneracies for the open-string states can be read from the coefficients in the power-series expansion of the generating function, Eq. (14.63):

$$f_{os}(x) = \frac{1}{x} + 24 + 324x + 3200x^2 + 25650x^3 + 176256x^4 + \dots$$

The degeneracies of the first five levels are therefore 1, 24, 325, 3,200, and 25,650. Closed-string states can be constructed by combining any allowed left-moving excitation with the correct value of $\alpha' M^2$ with any allowed right-moving excitation with the correct value of $\alpha' M^2$, so the number of combinations is given in the following table:

Closed Bosonic String	
$\frac{1}{2}\alpha' M^2$	Degeneracy
-2	$(1)^2 = 1$
0	$(24)^2 = 576$
2	$(324)^2 = 104,976$
4	$(3,200)^2 = 10,240,000$
6	$(25,650)^2 = 657,922,500$

Table 1

- (b) For Type II closed superstrings, we first list the M^2 eigenvalues and degeneracies for the (NS+,NS+) sector, which are bosons. The degeneracies can be found by expanding $f_{NS+}(x)$ as defined by Eq. (14.73),

$$\bar{f}_{NS}(x) = \frac{1}{2\sqrt{x}} \left[\prod_{n=1}^{\infty} \left(\frac{1+x^{n-\frac{1}{2}}}{1-x^n} \right)^8 - \prod_{n=1}^{\infty} \left(\frac{1-x^{n-\frac{1}{2}}}{1-x^n} \right)^8 \right],$$

which should give precisely the terms of Eq. (14.67) which are even in powers of $\sqrt{x}$. The only problem is that Eq. (14.67) does not show enough terms for our current purposes. The most straightforward solution to this shortcoming is to use a symbolic manipulation program to expand it further. I find, using a program called *Derive*, that

$$\begin{aligned} f_{NS}(x) &= \frac{1}{\sqrt{x}} \prod_{n=1}^{\infty} \left(\frac{1+x^{n-\frac{1}{2}}}{1-x^n} \right)^8 \\ &= \frac{1}{\sqrt{x}} + 8 + 36\sqrt{x} + 128x + 402x\sqrt{x} + 1152x^2 + 3064x^2\sqrt{x} + \\ &\quad + 7680x^3 + 18351x^3\sqrt{x} + 42112x^4 + 93300x^4\sqrt{x} + 200448x^5 + \dots, \end{aligned}$$

which is more than we need. Alternatively, one could trust the statements that have been made about supersymmetry and the matching of bosonic and fermionic levels, in which case the necessary coefficients can be found in Eq. (14.71). The first 5 values of $\alpha'M^2$ for the truncated Neveu-Schwarz open-string spectrum are then 0, 1, 2, 3, and 4. Taking into account the fact that any left-moving excitation of the correct level can be combined with any right-moving excitation of the correct level, the table is as follows:

(NS+, NS+) Sector, Closed String	
$\frac{1}{2}\alpha'M^2$	Degeneracy
0	$(8)^2 = 64$
2	$(128)^2 = 16,384$
4	$(1,152)^2 = 1,327,104$
6	$(7,680)^2 = 58,982,400$
8	$(42,112)^2 = 1,773,420,540$

Table 2

These are spacetime bosons. Next we have the fermionic states of the (NS+,R+) and (R-,NS+) sectors of Type IIA strings, or the (NS+,R-) and (R-,NS+) sectors of the Type IIB strings. Finally we would have the other spacetime bosons, which would be an (R-,R+) sector for Type IIA strings or an (R-,R-) sector for Type IIB strings. The counting is the same for each of these sectors in both Type IIA and IIB. For any one of these sectors, the new ingredient is the counting for $R+$ or $R-$ states. According to Eq. (14.70), however,

$$\begin{aligned} f_{R\pm}(x) &= 8 \prod_{n=1}^{\infty} \left(\frac{1+x^n}{1-x^n} \right)^8 \\ &= 8 + 128x + 1152x^2 + 7680x^3 + 42112x^4 + \dots \quad , \end{aligned}$$

which shows that the coefficients are identical to those of the NS+ sector. This equality is essential for the supersymmetry to hold, but it also means that each of the four sectors for both Type IIA and Type IIB closed superstrings has the same degeneracy table as the one shown as Table 2. Since each theory contains two sectors of spacetime bosons and two sectors of spacetime fermions, the number of spacetime bosons at any mass level will be twice the number shown in Table 2, and the number of bosons will be the same.

Problem 14.5: Counting states in heterotic $SO(32)$ string theory

(a) The normal ordering constant is

$$a = 8a_B + 32a_{NS} = -\frac{8}{24} - \frac{32}{48} = -\frac{1}{3} - \frac{2}{3} = -1 .$$

So, for the left NS' sector,

$$\alpha' M_L^2 = \sum_{n=1}^{\infty} \bar{\alpha}_{-n}^I \bar{\alpha}_n^I + \sum_{r=\frac{1}{2}, \frac{3}{2}, \dots} r \lambda_{-r}^A \lambda_r^A - 1 = N^\perp - 1 .$$

The GSO projection keeps the ground state, so it then implies that the fermionic operators λ_n^A must be applied in pairs. Thus, the lowest states of the left NS' + sector are:

$\alpha' M_L^2$	States	Number of States	Total Number of States
-1	$ \text{NS}'\rangle_L$	1	1
0	$\bar{\alpha}_{-1}^I \text{NS}'\rangle_L$	8	504
	$\lambda_{-\frac{1}{2}}^A \lambda_{-\frac{1}{2}}^B \text{NS}'\rangle_L$ (antisymmetric in A and B)	$\frac{32 \cdot 31}{2} = 496$	
1	$\bar{\alpha}_{-2}^I \text{NS}'\rangle_L$	8	40,996
	$\bar{\alpha}_{-1}^I \bar{\alpha}_{-1}^J \text{NS}'\rangle_L$ (symmetric in I and J)	$\frac{8 \cdot 9}{2} = 36$	
	$\bar{\alpha}_{-1}^I \lambda_{-\frac{1}{2}}^A \lambda_{-\frac{1}{2}}^B \text{NS}'\rangle_L$ (antisymmetric in A and B)	$8 \times 496 = 3,968$	
	$\lambda_{-\frac{1}{2}}^A \lambda_{-\frac{1}{2}}^B \lambda_{-\frac{1}{2}}^C \lambda_{-\frac{1}{2}}^D \text{NS}'\rangle_L$ (antisymmetric in $A, B, C,$ and D)	$\frac{32 \cdot 31 \cdot 30 \cdot 29}{1 \cdot 2 \cdot 3 \cdot 4} = 35,960$	
	$\lambda_{-\frac{3}{2}}^A \lambda_{-\frac{1}{2}}^B \text{NS}'\rangle_L$	$32 \times 32 = 1,024$	

(b) The normal ordering constant is

$$a = 8 a_B + 32 a_R = -\frac{8}{24} + \frac{32}{24} = 1 .$$

Then, for the left R' sector,

$$\alpha' M_L^2 = \sum_{n=1}^{\infty} (\bar{\alpha}_{-n}^I \bar{\alpha}_n^I + n \lambda_{-n}^A \lambda_n^A) + 1 = N^\perp + 1 .$$

The GSO projection keeps all of the $R'+$ ground states, $|R_\alpha\rangle_L$, and discards the $R'-$ ground states, $|R_{\bar{\alpha}}\rangle_L$. The application of a fermionic operator reverses the fermion number $(-1)^{F_L}$, so states with an odd number of fermionic creation operators must be built on $|R_{\bar{\alpha}}\rangle_L$ vacua, and states with an even number of fermionic creation operators must be built on $|R_\alpha\rangle_L$ vacua. 16 of the λ_0^A are creation operators, creating 2^{16} vacua, half with $(-1)^{F_L} = 1$ and half with $(-1)^{F_L} = -1$. The lowest states of the left $R'+$ sector are therefore:

$\alpha' M_L^2$	States	Number of States	Total Number of States
1	$ R_\alpha\rangle_L$	2^{15}	32,768
2	$\bar{\alpha}_{-1}^I R_\alpha\rangle_L$	$8 \times 2^{15} = 262,144$	1,310,720
	$\lambda_{-1}^A R_{\bar{\alpha}}\rangle_L$	$32 \times 2^{15} = 1,048,576$	

(c) A tachyonic heterotic string state would require both a tachyonic left-moving state and a tachyonic right-moving state with the same value of M^2 . There is a left-moving tachyon, but there is no right-moving tachyon. Indeed, we were told that the right moving part of the theory is an open superstring, with GSO projection, so that it is comprised by NS+ and R- sectors, and there is no tachyon here. So there are no tachyonic string states in the heterotic theory.

Recalling that the right-moving massless states are eight states $b_{-1/2}^I |\text{NS}\rangle_R$ in the NS sector and eight states $|\text{R}_a\rangle_R$ in the R sector, we get the massless heterotic states by tensoring with the massless states in the left sector, which only occur in the GSO projected NS' sector (see part (a)). We thus find the following heterotic massless states:

$\bar{\alpha}_{-1}^I \text{NS}'\rangle_L \text{R}_a\rangle_R \text{ gives } 8 \times 8 = 64 \text{ fermionic particle states}$ $\lambda_{-\frac{1}{2}}^A \lambda_{-\frac{1}{2}}^B \text{NS}'\rangle_L \text{R}_a\rangle_R \text{ gives } 496 \times 8 = 3,968 \text{ fermionic particle states}$ $\bar{\alpha}_{-1}^I \text{NS}'\rangle_L b_{-\frac{1}{2}}^J \text{NS}\rangle_R \text{ gives } 8 \times 8 = 64 \text{ bosonic particle states:}$ a traceless symmetric graviton (35 components), an antisymmetric Kalb-Ramond field (28 components), and a scalar, the dilaton. $\lambda_{-\frac{1}{2}}^A \lambda_{-\frac{1}{2}}^B \text{NS}'\rangle_L b_{-\frac{1}{2}}^I \text{NS}\rangle_R \text{ gives } 496 \times 8 = 3,968 \text{ bosonic particle states,}$ consisting of 496 vector fields. (These are, in fact, the gauge fields of SO(32).)

As stated in the problem, for $\alpha' M^2 = 4$, the left- and right-moving states must each have $\alpha' M^2 = 1$.

With $\alpha' M^2 = 1$ there are 40,996 left-moving NS'+ states and 32,768 left-moving R'+, for a total of 73,764 left-moving states at this level.

There are 128 right-moving R- states at $\alpha' M^2 = 1$ ($\alpha_{-1}^I |\text{R}_a\rangle_R$ and $d_{-1}^I |\text{R}_{\bar{a}}\rangle_R$). By supersymmetry, there are also 128 right-moving NS+ states, for a total of 256 right-moving states.

The total number of string states is then found by multiplication of the two numbers above: $73,764 \times 256 = 18,883,584$ heterotic string states with $\alpha' M^2 = 4$.

(d) The $\alpha' M^2$ values for left NS' states are given by the equation in part (a),

$$\alpha' M_L^2 = \sum_{n=1}^{\infty} \bar{\alpha}_{-n}^I \bar{\alpha}_n^I + \sum_{r=\frac{1}{2}, \frac{3}{2}, \dots} r \lambda_{-r}^A \lambda_r^A - 1 .$$

This has the same form as the standard NS open-string expression, except that the fermionic oscillator index A takes on 32 different values, instead of 8, and the normal-ordering constant has the value -1 instead of -1/2. We thus write, before GSO projection

$$f_{\text{NS}'}(x) = \frac{1}{x} \prod_{n=1}^{\infty} \frac{\left(1 + x^{n-\frac{1}{2}}\right)^{32}}{(1 - x^n)^8} .$$

To enforce the GSO projection we add to $f_{\text{NS}' +}(x)$ a copy of itself except that we change the sign of terms that arise with an odd number of fermions. This is done by changing the sign in the numerator factors. The result is then divided by two, so we get

$$f_{\text{NS}' +}(x) = \frac{1}{2x} \left[\prod_{n=1}^{\infty} \frac{(1 + x^{n-\frac{1}{2}})^{32}}{(1 - x^n)^8} + \prod_{n=1}^{\infty} \frac{(1 - x^{n-\frac{1}{2}})^{32}}{(1 - x^n)^8} \right] .$$

For the left R' sector, from (b) we have

$$\alpha' M_L^2 = \sum_{n=1}^{\infty} (\bar{\alpha}_{-n}^I \bar{\alpha}_n^I + n \lambda_{-n}^A \lambda_n^A) + 1 .$$

This formula differs from the standard one by having a fermionic oscillator index A that takes on 32 values instead of 8, and also by having a normal-ordering constant equal to 1, instead of 0. The power of the numerator in the standard expression is therefore changed from 8 to 32, and the prefactor is multiplied by $x^1 = x$. In addition, the number of vacuum states is 2^{15} instead of $2^3 = 8$, so the factor of 8 in the prefactor is replaced by 2^{15} :

$$f_{\text{R}' +}(x) = 2^{15} x \prod_{n=1}^{\infty} \frac{(1 + x^n)^{32}}{(1 - x^n)^8} .$$

The full generating function $f_L(x)$ for left-moving states is then given by adding these two functions, so that each coefficient in the power series is the sum of the coefficients from the two functions: $f_L(x) = f_{\text{NS}' +}(x) + f_{\text{R}' +}(x)$. Therefore

$$f_L(x) = \frac{1}{2x} \left[\prod_{n=1}^{\infty} \frac{(1 + x^{n-\frac{1}{2}})^{32}}{(1 - x^n)^8} + \prod_{n=1}^{\infty} \frac{(1 - x^{n-\frac{1}{2}})^{32}}{(1 - x^n)^8} \right] + 2^{15} x \prod_{n=1}^{\infty} \frac{(1 + x^n)^{32}}{(1 - x^n)^8} .$$

An expansion of this with Mathematica gives:

$$f_L(x) = \frac{1}{x} + 504 + 73764 x + 2695040 x^2 + 54755730 x^3 + \dots$$

Here we see the tachyon, unmatched in the right-moving sector, the 504 states that come from the $\text{NS}' +$ sector and the 73764 total states we found at $\alpha' M^2 = 1$ in the left-moving sector. Now we see that there are 2695040 states in the left sector at $\alpha' M^2 = 2$. On the right sector there are 1152×2 states with $\alpha' M^2 = 2$ (see (14.71) and double it). The total number of heterotic states with $\alpha' M^2 = 8$ is therefore

$$2695040 \times 1152 \times 2 = 6209372160 .$$